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Orthogonal Polynomials on Arcs of the Unit Circle II. Orthogonal Polynomials with Periodic Reflection Coefficients

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First we give necessary and sufficient conditions on a set of intervals $E_l = \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}]$, $\varphi_1 < \cdots < \varphi_{2l}$ and $\varphi_{2l} - \varphi_1 \leqslant 2\pi$, such that on E_l there exists a real trigonometric polynomial $\tau_N(\varphi)$ with maximal number, i.e., $N + l$, of extremal points on E_l. The associated algebraic polynomial $\mathscr{T}_N(z) = z^{N/2}\tau_N(z)$, $z = e^{i\varphi}$, is called the complex Chebyshev polynomial. Then it is shown that polynomials orthogonal on E_l have periodic reflection coefficients if and only if they are orthogonal on E_l with respect to a measure of the form $\sqrt{-\prod_{j=1}^{2l} \sin((\varphi - \varphi_j)/2)/\mathscr{A}(\varphi)}\, d\varphi +$ certain point measures, where $\mathscr{A}$ is a real trigonometric polynomial with no zeros on E_l and there exists a complex Chebyshev polynomial on E_l. Let us point out in this connection that Geronimus has shown that orthogonal polynomials generated by periodic reflection coefficients of absolute value less than 1 are orthogonal with respect to a measure of the above type. Furthermore, we derive explicit representations of the corresponding orthogonal polynomials with the help of the complex Chebyshev polynomials. Finally, we provide a characterization of those definite functionals to which orthogonal polynomials with periodic reflection coefficients of modulus unequal to one are orthogonal.

1. NOTATION

Let (a_n) be a given sequence of complex numbers with $|a_n| \neq 1$ for $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and let a monic sequence of polynomials $P_n(z) = z^n + \cdots$ be generated by the recurrence relation

$$P_{n+1}(z) = zP_n(z) - \bar{a}_n P_n^*(z), \tag{1.1}$$

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where $P_n^*(z) := z^n \overline{P}_n(1/z)$ denotes the reciprocal polynomial of P_n. Then it is known that the polynomials P_n are orthogonal with respect to a definite linear functional $\mathscr{L}$, i.e.,

$$\mathscr{L}(x^{-j}P_n) = 0 \qquad \text{for} \quad j = 0, ..., n-1, \tag{1.2}$$

where $\mathscr{L}$ is defined on the set of Laurent polynomials and has the property that $\overline{\mathscr{L}(x^{-j})} = \mathscr{L}(x^j)$ for $j \in \mathbb{N}_0$. Let us recall that by the definiteness the moments

$$c_j := \mathscr{L}(x^{-j}), \qquad j \in \mathbb{Z}, \tag{1.3}$$

satisfy the condition

$$\det \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{pmatrix} \neq 0 \qquad \text{for all} \quad n \in \mathbb{N}_0. \tag{1.4}$$

Conversely, if a definite functional $\mathscr{L}$ is given whose moments c_j satisfy the conditions that $c_{-j} = \bar{c}_j \in \mathbb{C}$ and $\sum_{j=0}^{\infty} c_j z^j$ converges in a neighbourhood of $z = 0$, then note that in this case

$$\mathscr{L}\left(\frac{x+z}{x-z}\right) = c_0 + 2 \sum_{j=1}^{\infty} c_j z^j,$$

then there exists a uniquely determined sequence of monic polynomials $P_n(z) = z^n + \cdots$ such that (1.2) holds and the orthogonal polynomials P_n satisfy a recurrence relation of the form (1.1). The so-called Schur parameters or reflection coefficients (a_n) are given by the relation

$$a_n = \frac{\overline{\mathscr{L}(xP_n)}}{\mathscr{L}(x^{-n}P_n)} = -\overline{P_{n+1}(0)} \in \mathbb{C}, \tag{1.5}$$

which have, by the definiteness of $\mathscr{L}$, the property that $|a_n| \neq 1$ for all $n \in \mathbb{N}_0$.

Next let us assume that the given sequence of complex numbers (a_n) satisfy the stronger condition

$$|a_n| < 1 \qquad \text{for} \quad n \in \mathbb{N}_0, \tag{1.6}$$

which is equivalent to the fact that the determinants in (1.4) are all greater than zero (see e.g. [9, p. 5]). Then it is known that the orthogonal polynomials are even orthogonal with respect to a distribution function σ (as usual, a function σ is called the distribution function if it is real, bounded, and nondecreasing with an infinite set of points of increase), i.e., (1.2) takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ij\varphi} P_n(e^{i\varphi})\, d\sigma(\varphi) = 0 \qquad \text{for} \quad j = 0, ..., n-1. \tag{1.7}$$

Let us note that the Stieltjes transform F defined by

$$\begin{aligned} F(z) &:= \frac{1}{2\pi c_0}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\sigma(\varphi) = \frac{1}{c_0}\left(c_0 + 2\sum_{j=1}^{\infty} c_j z^j\right) \\ &= \frac{1}{c_0}\,\mathscr{L}\left(\frac{x+z}{x-z}\right), \qquad |z|<1, \end{aligned} \tag{1.8}$$

is a Carathéodory function, abbreviated as C-function, i.e., F is analytic on $|z|<1$ with positive real part there and $F(0)=1$.

In what follows the polynomials of the second kind of a polynomial Q with respect to a linear functional $\mathscr{L}$, defined by

$$\mathscr{L}\left(\frac{x+z}{x-z}\,(Q(x)-Q(z))\right) \qquad \text{if} \quad \partial Q > 0$$

$$\text{and} \qquad \mathscr{L}(1)\, Q(0) \qquad \text{if} \quad \partial Q = 0,$$

where $\mathscr{L}$ acts on x, will play an important role, in particular those monic polynomials of the second kind corresponding to P_n orthogonal with respect to the definite functional $\mathscr{L}$, which are denoted by

$$\Omega_n(z) := \begin{cases} \dfrac{1}{c_0}\,\mathscr{L}\left(\dfrac{x+z}{x-z}\,(P_n(x)-P_n(z))\right), & \text{if } n\in\mathbb{N} \\ 1, & \text{if } n=0, \end{cases} \qquad c_0 = \mathscr{L}(1) \tag{1.9}$$

Note that by the definiteness of the functional $\mathscr{L}$ the polynomial Ω_n is of exact degree n. Let us point out that, in view of (1.1) and (1.9), the Ω_n's satisfy the recurrence relation

$$\Omega_{n+1}(z) = z\Omega_n(z) + \bar{a}_n \Omega_n^*(z), \qquad n\in\mathbb{N}_0, \tag{1.10}$$

i.e., the reflection coefficients a_n in (1.1) are replaced by $-a_n$. Furthermore the P_n's and Ω_n's are related by

$$P_n^*(z)\,\Omega_n(z) + P_n(z)\,\Omega_n^*(z) = 2d_n z^n \qquad \text{with} \quad d_n = \prod_{j=0}^{n-1}(1-|a_j|^2). \tag{1.11}$$

Finally, let us also introduce the ν th, $\nu \in \mathbb{N}_0$, associated monic polynomials $(P_n^{(\nu)})$ and $(\Omega_n^{(\nu)})$ of (P_n) and (Ω_n), respectively, with respect to the definite linear functional $\mathscr{L}$ by the following shifted recurrence relation (1.1), resp. (1.10),

$$P_{n+1}^{(\nu)}(z) = zP_n^{(\nu)}(z) - \bar{a}_{n+\nu} P_n^{(\nu)*}(z), \qquad n \in \mathbb{N}_0, \qquad P_0^{(\nu)}(z) = 1 \tag{1.12}$$

$$\Omega_{n+1}^{(\nu)}(z) = z\Omega_n^{(\nu)}(z) + \bar{a}_{n+\nu} \Omega_n^{(\nu)*}(z), \qquad n \in \mathbb{N}_0, \qquad \Omega_0^{(\nu)}(z) = 1, \tag{1.13}$$

where the a_n's are the reflection coefficients from (1.1) corresponding to $\mathscr{L}$. We will denote $P_n^{(0)}$ by P_n and $\Omega_n^{(0)}$ by Ω_n. These associated polynomials have been investigated in detail by the first author in [17]. For the following we will need the fact that the Carathéodory function associated with $(P_n^{(\nu)})$ and denoted by $F^{(\nu)}$ has a representation of the form (cf. [17, Theorem 3.1]; see also [9, Theorem 18.2])

$$F^{(\nu)}(z) = \frac{F(z)(P_\nu(z) + P_\nu^*(z)) + (\Omega_\nu(z) - \Omega_\nu^*(z))}{F(z)(P_\nu(z) - P_\nu^*(z)) + (\Omega_\nu(z) + \Omega_\nu^*(z))}, \qquad F^{(\nu)}(0) = 1. \tag{1.14}$$

Let us give some relations between the associated polynomials and the original polynomials from (1.1) and (1.10), which can be seen by simple induction-arguments (for the positive-definite case, i.e., $|a_n| < 1$, these identities have been first shown in [17, Theorem 3.1 and Corollary 3.1]),

$$2P_{n+\nu} = (P_\nu + P_\nu^*)\, P_n^{(\nu)} + (P_\nu - P_\nu^*)\, \Omega_n^{(\nu)} \tag{1.15}$$

$$2\Omega_{n+\nu} = (\Omega_\nu - \Omega_\nu^*)\, P_n^{(\nu)} + (\Omega_\nu + \Omega_\nu^*)\, \Omega_n^{(\nu)} \tag{1.16}$$

$$2P_n^{(\nu)} = \frac{1}{d_\nu z^\nu} [P_{n+\nu}(\Omega_\nu + \Omega_\nu^*) - \Omega_{n+\nu}(P_\nu - P_\nu^*)] \tag{1.17}$$

$$2\Omega_n^{(\nu)} = \frac{1}{d_\nu z^\nu} [\Omega_{n+\nu}(P_\nu + P_\nu^*) - P_{n+\nu}(\Omega_\nu - \Omega_\nu^*)], \tag{1.18}$$

where $n, \nu \in \mathbb{N}_0$ and where d_ν is from (1.11).

In addition we will need the following notation. We write $H(z) = O(z^m)$, $m \in \mathbb{N}_0$, if H is analytic at $z = 0$ with a series expansion at $z = 0$ of the form $H(z) = \sum_{j=m}^{\infty} h_j z^j$. If $h_m \neq 0$ we write $H(z) = \dot{O}(z^m)$.

Further, if Q is a polynomial of exact degree $\partial Q \leqslant n$, we define the *modified reciprocal* polynomial $Q_n^{(*)}$ by

$$Q_n^{(*)}(z) := z^n \bar{Q}\left(\frac{1}{z}\right) = z^{n - \partial Q} Q^*(z). \tag{1.19}$$

Note that the exponent n of z in (1.19) is equal to the subindex on the left-hand side and that for the modified reciprocal polynomial the index n must be written explicitly. The reason why we distinguish between the modified reciprocal polynomial $Q_n^{(*)}$ and the reciprocal polynomial Q^* is that $Q_n^{(*)}(0)=0$ is possible (if $\partial Q<n$) whereas $Q^*(0)$ is always different from zero.

We call a polynomial Q of degree $\partial Q \leqslant n$ *selfreciprocal* if it satisfies

$$Q=\mu Q^*, \quad \text{resp., } Q=\mu Q_n^{(*)}, \qquad \text{where} \quad |\mu|=1.$$

Finally, we denote the space of complex algebraic polynomials of degree $\leqslant n$ by $\mathbb{P}_n$ and the space of real trigonometric polynomials of (integer or half-integer) degree $\leqslant n/2$ by

$$\Pi_{n/2}:=\left\{\sum_{k=0}^{\lfloor n/2 \rfloor} a_k \cos\left(\frac{n-2k}{2}\varphi\right)+b_k \sin\left(\frac{n-2k}{2}\varphi\right): a_k, b_k \in \mathbb{R}\right\}.$$

We say $\mathscr{D} \in \Pi_{n/2}$ is of exact degree $\partial \mathscr{D}=n/2$, if $|a_0|+|b_0| \neq 0$. As usual let $\mathbb{P}$ denote the set of all algebraic and Π the set of all real trigonometric polynomials. The polynomials from $\mathbb{P}$ and Π are assigned to each other by the following well known relation: If D is a selfreciprocal algebraic polynomial then

$$\mathscr{D}(\varphi):=e^{-i(\partial D/2)\varphi} \mu^{1/2} D(e^{i\varphi})$$

is a real trigonometric polynomial of degree $\partial \mathscr{D}=\partial D/2$ and vice versa.

2. PRELIMINARY CONSIDERATIONS AND FURTHER NOTATIONS

Polynomials orthogonal on the unit circle having reflection coefficients (a_n) with

$$\lim_{n \to \infty} a_n=0, \tag{2.1}$$

in particular, those satisfying the stronger Szegő condition

$$\sum_{n=0}^{\infty} |a_n|^2<\infty,$$

have been investigated in detail, see e.g. [9, 24]. Let us note that (2.1) corresponds, roughly speaking, to the case that the measure is supported on $[0, 2\pi]$. For the case of one interval of the form $[\alpha, 2\pi-\alpha]$ which corresponds, again roughly speaking, to the condition

$$\lim_{n \to \infty} a_n=a, \qquad \text{where} \quad |a|<1, \tag{2.2}$$

information about the behavior of the orthogonal polynomials, the measure, etc., is also available (see Akhiezer [1], Geronimus [6], and Golinskii, *et al.* [11]). Naturally the question arises as to what can be said on the reflection coefficients and the behavior of the orthogonal polynomials if the measure is supported on several intervals or, taking a look at (2.2), what can be said on the orthogonality measure if the reflection coefficients are asymptotically Nth periodic, i.e.,

$$\lim_{n\to\infty} a_{nN+j} = \tilde{a}_j \qquad \text{for} \quad j=0, ..., N-1,$$
$$\text{where} \quad |\tilde{a}|<1 \qquad \text{for} \quad j=0, ..., N-1, \tag{2.3}$$

and whether these two questions are related to each other. To be able to handle these questions we first have to understand what is going on in the simplest case, i.e., when the reflection coefficients are purely periodic from a certain index onward,

$$a_{n+N} = a_n \qquad \text{for} \quad n \geqslant n_0 \qquad \text{with} \quad |a_n|<1 \qquad \text{for} \quad n\in\mathbb{N}$$
$$\text{and} \qquad a_{n_0-1+N} \neq a_{n_0-1}. \tag{2.4}$$

Using continued fractions, Geronimus [8] determined the absolutely continuous part of the measure with respect to which polynomials with periodic reflection coefficients $0<|a_n|<1$ are orthogonal. Let us briefly demonstrate how to get Geronimus' results [8] in a simple way without using continued fractions. Let (a_n) be a sequence satisfying (2.4), let (P_n) be the monic orthogonal polynomials, and let F be the C-function associated with (a_n). Since by (1.12), (1.13), and property (2.4)

$$P_j^{(n)} \equiv P_j^{(n+N)} \quad \text{and} \quad \Omega_j^{(n)} \equiv \Omega_j^{(n+N)} \quad \text{for all} \quad j\in\mathbb{N} \quad \text{and all} \quad n\geqslant n_0,$$

we have by (1.14)

$$F^{(n)} \equiv F^{(n+N)} \qquad \text{for} \quad n\geqslant n_0$$

or written down explicitly,

$$\frac{F(P_n+P_n^*)+(\Omega_n-\Omega_n^*)}{F(P_n-P_n^*)+(\Omega_n+\Omega_n^*)} = \frac{F(P_{n+N}+P_{n+N}^*)+(\Omega_{n+N}-\Omega_{n+N}^*)}{F(P_{n+N}-P_{n+N}^*)+(\Omega_{n+N}+\Omega_{n+N}^*)}.$$

Solving this equation yields

$$F_{1,2} = \frac{\pm\widetilde{B}_{(n)}+\sqrt{\widetilde{R}_{(n)}}}{\pm\widetilde{A}_{(n)}}, \qquad n\geqslant n_0, \tag{2.5}$$

with

$$\tilde{A}_{(n)} = 2(P_n P^*_{n+N} - P^*_n P_{n+N})$$
$$\tilde{B}_{(n)} = P^*_n \Omega_{n+N} - \Omega^*_n P_{n+N} - \Omega_n P^*_{n+N} + P_n \Omega^*_{n+N} \tag{2.6}$$
$$\tilde{R}_{(n)} = \tilde{B}^2_{(n)} + 2\tilde{A}_{(n)}(\Omega_n \Omega^*_{n+N} - \Omega^*_n \Omega_{n+N}).$$

By forward calculation and using (1.1), (1.10), and (2.4) we get

$$\left.\begin{matrix}\tilde{A}_{(n)}\\ \tilde{B}_{(n)}\\ \tilde{C}_{(n)}\end{matrix}\right\} = z^{n-n_0} \prod_{j=n_0}^{n-1} (1-|a_j|^2) \cdot \left\{\begin{matrix}\tilde{A}_{(n_0)}\\ \tilde{B}_{(n_0)},\\ \tilde{C}_{(n_0)}\end{matrix}\right.$$
$$\tilde{R}_{(n)} = z^{2(n-n_0)} \prod_{j=n_0}^{n-1} (1-|a_j|^2)^2 \cdot \tilde{R}_{(n_0)}. \tag{2.7}$$

Thus we obtain by choosing the "positive" function in (2.5)

$$F = \frac{\tilde{B}_{(n_0)} + \sqrt{\tilde{R}_{(n_0)}}}{\tilde{A}_{(n_0)}}. \tag{2.8}$$

Furthermore, we have by (2.6) that $\tilde{A}_{(n_0)}$, $\tilde{B}_{(n_0)}$, and $\tilde{R}_{(n_0)}$ are selfreciprocal polynomials, where it can be shown that all the zeros of $\tilde{R}_{(n_0)}$ lie on the circumference $|z|=1$, and, what is important in what follows, that by the third relation in (2.6)

$$\tilde{B}_{(n_0)}(z_j) = \lambda_j \sqrt{\tilde{R}_{(n_0)}(z_j)} \tag{2.9}$$

at the zeros z_j of $\tilde{A}_{(n_0)}$, where $\lambda_j \in \{-1, +1\}$. For simplicity let us consider now the case $n_0 = 0$, i.e., the case when there is no preperiod. Instead of $\tilde{R}_{(0)}$, $\tilde{A}_{(0)}$, $\tilde{B}_{(0)}$ let us write $\tilde{R}$, $\tilde{A}$, $\tilde{B}$. By (2.6) we have

$$\tilde{R} = \tilde{R}^* \qquad \text{and} \qquad \tilde{A} = -\tilde{A}^*, \tag{2.10}$$

thus $e^{-iN\varphi}\tilde{R}(e^{i\varphi})$ and $ie^{-i(N/2)\varphi}\tilde{A}(e^{i\varphi})$ are real trigonometric polynomials. Let now $E_l := \{\varphi : e^{-iN\varphi}\tilde{R}(e^{i\varphi}) \leqslant 0\} =: \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}]$ be a subset of an interval of length 2π which consists of l, $l \leqslant N$, intervals. It can be shown (compare (1.7) and (2.13) below) that

$$ie^{-i(N/2)\varphi}\tilde{A}(e^{i\varphi}) = \prod_{k=1}^{N} \sin\left(\frac{\varphi - \xi_k}{2}\right) \in \Pi_{N/2}, \tag{2.11}$$

where there is an odd number (counted according to their multiplicities) of ξ_k's in each interval $[\varphi_{2j}, \varphi_{2j+1}]$, $j = 1, ..., l$, $\varphi_{2l+1} := \varphi_1 + 2\pi$. If we further assume that all the zeros of $\tilde{A}$ are simple we obtain by [19, Theorem 2.1])

$$\begin{aligned}\frac{\tilde{B}(z)+\sqrt{\tilde{R}(z)}}{\tilde{A}(z)} &= \sum_{j=1}^{l}(-1)^j \int_{\varphi_{2j-1}}^{\varphi_{2j}} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{\sqrt{-e^{-iN\varphi}\tilde{R}(e^{i\varphi})}}{ie^{-i(N/2)\varphi}\tilde{A}(e^{i\varphi})}\,d\varphi \\ &\quad -\frac{1}{2}\sum_{j=1}^{N}(1-\lambda_j)\frac{e^{i\xi_j}+z}{e^{i\xi_j}-z}e^{-i\xi_j}\frac{\sqrt{\tilde{R}(e^{i\xi_j})}}{\tilde{A}'(e^{i\xi_j})} \\ &= \int_{E_l}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\left|\frac{\sqrt{\tilde{R}(e^{i\varphi})}}{\tilde{A}(e^{i\varphi})}\right| d\varphi - \frac{1}{2}\sum_{j=1}^{N}(1-\lambda_j) \\ &\quad \times \frac{e^{i\xi_j}+z}{e^{i\xi_j}-z}e^{-i\xi_j}\frac{\sqrt{\tilde{R}(e^{i\xi_j})}}{\tilde{A}'(e^{i\xi_j})}, \end{aligned} \tag{2.12}$$

where $\lambda_j \in \{-1, +1\}$ for $j=1, ..., N$ is given by (2.9). Let us point out that in the last equation we have used the fact that the real trigonometric polynomial $ie^{-i(N/2)\varphi}\tilde{A}(e^{i\varphi})$ changes sign from interval $(\varphi_{2j-1}, \varphi_{2j})$ to interval $(\varphi_{2j+1}, \varphi_{2j+2})$, $j=1, ..., l-1$. Furthermore, we see by (2.8) and (2.9) that an appearance of a point measure only depends on the fact of whether $\tilde{B}(e^{i\xi_j})$ interpolates $+\sqrt{\tilde{R}(e^{i\xi_j})}$ or $-\sqrt{\tilde{R}(e^{i\xi_j})}$. This appearance, resp. non-appearence, of a point measure becomes clear when we take a look at the left-hand side in (2.12) and recall that the first term in the last equality in (2.12) is a function analytic outside $\{e^{i\varphi}\colon \varphi \in E_l\}$.

An analog representation as in (2.12) also holds if $n_0 > 0$ or if $\tilde{A}_{(n_0)}$ has multiple zeros. Hence, the orthogonal polynomials with periodic reflection coefficients from a certain index n_0 onward are orthogonal with respect to a measure whose absolute continuous part is of the form

$$\begin{aligned}\left|\frac{\sqrt{\tilde{R}_{n_0}(z)}}{\tilde{A}_{n_0}(z)}\right| &= \frac{\sqrt{-e^{-i(N+2n_0)\varphi}\tilde{R}_{n_0}(e^{i\varphi})}}{|ie^{-i((N/2)+n_0)\varphi}\tilde{A}_{n_0}(e^{i\varphi})|} \\ &=: \frac{\sqrt{-\tilde{\mathscr{R}}(\varphi)}}{|\tilde{\mathscr{A}}(\varphi)|} \qquad \text{for} \quad \varphi \in \operatorname{Int}(E_l), \end{aligned} \tag{2.13}$$

and 0 otherwise, where $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{A}}$ are real trigonometric polynomials. Since the function in (2.13) is integrable it can be derived that $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{A}}$ have to be of the form

$$\tilde{\mathscr{R}}(\varphi) = u^2(\varphi)\,\mathscr{V}(\varphi)\,\mathscr{W}(\varphi) \qquad \text{and} \qquad \tilde{\mathscr{A}}(\varphi) = u(\varphi)\,\mathscr{V}(\varphi)\,\mathscr{A}(\varphi),$$

$$\text{with} \quad u, \mathscr{V}, \mathscr{W}, \mathscr{A} \in \Pi,$$

where u vanishes exactly in those points from $\operatorname{Int}(E_l)$ where $\tilde{\mathscr{R}}$ vanishes (note that the weight function in (2.13) does not change sign), and where the zeros of $\mathscr{V}$, which have to be boundary points of E_l, are the common zeros of $\mathscr{R} := \mathscr{V} \cdot \mathscr{W}$ and $\tilde{\mathscr{A}}$. Thus the weight function (2.13) takes the form

$$\sqrt{-\frac{\mathscr{W}(\varphi)}{\mathscr{V}(\varphi)}}\frac{1}{|\mathscr{A}(\varphi)|} = \left|\frac{\mathscr{W}(\varphi)}{\mathscr{A}(\varphi)\sqrt{\mathscr{R}(\varphi)}}\right| \qquad \text{for} \quad \varphi \in E_l \text{ and zero otherwise,} \tag{2.14}$$

where $\mathscr{A}$ has now no zero on E_l and where (as we will see e.g. in Lemma 4.2) $\mathscr{R} \in \Pi_l$.

What about the converse direction, i.e., do polynomials which are orthogonal on several arcs with respect to a weight function of the form (2.14) have periodic reflection coefficients? It will turn out that this does not hold in general and that this periodic property depends only on the set E_l. The characterization of such sets of intervals E_l, which is carried out in Section 3, will be a main point of this paper. For the description we introduce the so-called complex Chebyshev polynomials (abbreviated as complex T-polynomials), which are closely related to trigonometric polynomials which deviate least from zero on E_l with respect to the maximum-norm and have maximal number of equi-oscillation points on E_l. These complex T-polynomials enable us to give a simple description of those intervals where polynomials orthogonal with respect to a weight function of the form (2.14) have periodic reflection coefficients. Why is this important? It is important in the description of the asymptotic behaviour of general orthogonal polynomials on such sets E_l. Indeed, since the degree of the polynomial $\tilde{A}_{(n_0)}$ in (2.8), resp. the degree of $\mathscr{A}$ in (2.14), is arbitrary we shall expect that all weight functions on E_l which can be approximated sufficiently well by weight functions of the form (2.14) will have reflection coefficients which are not pure but are asymptotically periodic. Further, we shall expect that the asymptotic behavior of these orthogonal polynomials can be described with the help of those polynomials orthogonal with respect to a weight function of the form (2.14). For those last-mentioned polynomials, a simple closed formula will be derived in Section 4. The above-described application to asymptotic representations of orthogonal polynomials with asymptotically periodic reflection coefficients will be given in a forthcoming paper [21]. For polynomials orthogonal on the real line the corresponding questions above have been investigated by the first author in [15].

Let us also point out that the asymptotic behavior of polynomials orthogonal on very general sets of the complex plane as on arcs and Jordan curves have been studied by Widom [25]. The problem is however that if Widom's nice asymptotic formula is not entirely explicit, it is assumed that the solution of a certain so-called Jacobi inversion problem is known. Aptekarev [3] tried to overcome this difficulty by using

Riemann theta functions. But still it is often difficult to use and work with these asymptotic formulas. Nevertheless, as one of the referees pointed out, it might be possible to prove with the help of Widom's result (compare with Aptekarev's idea in [3] and see also [16, Remark 2.6(a) and Corollary 3.2]) that the reflection coefficients are pseudo-periodic if and only if the harmonic measure at ∞ of each arc is a rational number. Finally, let us note that our approach, developed in [19, 21], is completely different from those in [3, 25].

Since we want to characterize all functionals which generate orthogonal polynomials with periodic reflection coefficients (a_n), not only those with $|a_n| < 1$, we have to consider the functionals $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ defined in (2.26) below, which were already investigated by the authors in [19]. By the way, to get a characterization of orthogonal polynomials with respect to weight functions of the form (2.14) and eventually a point measure at the zeros of $\mathscr{A}$, it is simpler from a technical point of view to get a characterization of the orthogonal polynomials with respect to the more complicated looking functions $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$. For the meaning of the following somewhat voluminous notations one should keep in mind the above-derived measures, see (2.11)–(2.14), to which polynomials with periodic reflection coefficients are orthogonal. Hence, in correspondence with the above considerations, let us set

$$\begin{aligned} E_l &:= \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}], \qquad \operatorname{Int}(E_l) := \bigcup_{j=1}^{l} (\varphi_{2j-1}, \varphi_{2j}), \\ \Gamma_{E_l} &:= \{e^{i\varphi} : \varphi \in E_l\}, \end{aligned} \tag{2.15}$$

where $\varphi_1 < \varphi_2 < \cdots < \varphi_{2l}$ and $E_l \subseteq [a, a+2\pi)$ for an adequate $a \in \mathbb{R}$, i.e., $\varphi_{2l} - \varphi_1 < 2\pi$. To the set E_l we assign the real trigonometric polynomial $\mathscr{R}$ by

$$\mathscr{R}(\varphi) := 4^l \prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right), \tag{2.16}$$

which has only simple zeros and vanishes exactly at the boundary points of E_l. Further, it satisfies

$$\mathscr{R}(\varphi) < 0 \qquad \text{on} \quad \operatorname{Int}(E_l). \tag{2.17}$$

As it will turn out, it is helpful to write $\mathscr{R}$ as

$$\mathscr{R}(\varphi) = \mathscr{V}(\varphi)\, \mathscr{W}(\varphi), \qquad \mathscr{V}, \mathscr{W} \in \Pi \quad \text{with } \partial\mathscr{V} =: v,\ \partial\mathscr{W} =: w. \tag{2.18}$$

Finally, let $\mathscr{A} \in \Pi$ be an arbitrary real trigonometric polynomial which has no zeros on E_l and whose degree $\partial\mathscr{A} =: a$ satisfies

$$a \geqslant w - \frac{l}{2} \qquad \text{and} \qquad \left(a + \frac{l}{2} - w\right) \text{ is an integer.} \tag{2.19}$$

In what follows we sometimes need the explicit representation of $\mathscr{A}$, i.e.,

$$\mathscr{A}(\varphi) = c_{\mathscr{A}} \prod_{j=1}^{m^*} \left(\sin\left(\frac{\varphi - \xi_j}{2}\right)\right)^{m_j}, \qquad c_{\mathscr{A}} \in \mathbb{R}\backslash\{0\}, \quad \xi_j \notin E_j, \quad \sum_{j=1}^{m^*} m_j = 2a. \tag{2.20}$$

Note that the ξ_j's are either real or they appear in pairs of complex conjugate numbers, since $\mathscr{A}$ is a real trigonometric polynomial.

If $\mathscr{W}(\varphi)/\sqrt{|\mathscr{R}(\varphi)|}$ is integrable on E_l, i.e., has poles at most of order $\frac{1}{2}$, then we define the integrable function

$$f(\varphi; \mathscr{A}, \mathscr{W}) := \begin{cases} \dfrac{\mathscr{W}(\varphi)}{\mathscr{A}(\varphi)\, r(\varphi)}, & \varphi \in E_l \\ 0, & \varphi \notin E_l, \end{cases} \tag{2.21}$$

where

$$\frac{1}{r(\varphi)} := \frac{(-1)^j}{\sqrt{|\mathscr{R}(\varphi)|}} \qquad \text{for} \quad \varphi \in (\varphi_{2j-1}, \varphi_{2j}), \quad j = 1, ..., l. \tag{2.22}$$

Since we have only assumed that $\mathscr{A}$ has no zero on E_l, the function $f(\varphi; \mathscr{A}, \mathscr{W})$ may have sign changes and is therefore in general no weight function in the classical sense. Let us consider for example $f(\varphi; 1, 1)$. Then the orthogonality condition of a polynomial $P_n(z)$ with respect to $f(\varphi; 1, 1)$ reads as

$$\sum_{j=1}^{l} (-1)^j \int_{\varphi_{2j-1}}^{\varphi_{2j}} e^{-ik\varphi} P_n(e^{i\varphi}) \frac{d\varphi}{\sqrt{|\mathscr{R}(\varphi)|}} = \int_{E_l} e^{-ik\varphi} P_n(e^{i\varphi}) \frac{1}{r(\varphi)}\, d\varphi = 0$$
$$\text{for} \quad k = -\tilde{m}, ..., \tilde{n} - 1,$$

where in view of the sign changes of $r(\varphi)$ it may happen that $\tilde{m} > 0$ and $\tilde{n} > n$. For further examples see [19]. But if $\operatorname{sgn} \mathscr{A}(\varphi) = (-1)^j$ on $[\varphi_{2j-1}, \varphi_{2j}]$ as in the derivation of (2.11)–(2.14), then $f(\varphi; \mathscr{A}, \mathscr{W})$ becomes a weight function in the classical sense and is of the form (2.14).

Let us now define the algebraic selfreciprocal polynomials R, V, W, and A assigned to the trigonometric polynomials $\mathscr{R}$, $\mathscr{V}$, $\mathscr{W}$, and $\mathscr{A}$ in the natural way by

$$\begin{aligned} R(e^{i\varphi}) &:= e^{il\varphi}\mathscr{R}(\varphi), \qquad \partial R = 2l, \qquad R = R^*, \\ R(0) &= (-1)^l \exp\left(\frac{i}{2}\sum_{j=1}^{2l} \varphi_j\right) \\ V(e^{i\varphi}) &:= e^{iv\varphi}\mathscr{V}(\varphi), \qquad \partial V = 2v, \qquad V = V^* \\ W(e^{i\varphi}) &:= e^{iw\varphi}\mathscr{W}(\varphi), \qquad \partial W = 2w, \qquad W = W^* \end{aligned} \tag{2.23}$$

and, by technical reasons (compare (2.10)),

$$A(e^{i\varphi}) := -ie^{ia\varphi}\mathscr{A}(\varphi), \qquad \partial A = 2a, \quad A = -A^*. \tag{2.24}$$

As in (2.20) let us give the explicit representation of A

$$A(z) = c_A \prod_{j=1}^{m^*} (z - z_j)^{m_j}, \qquad \text{where} \quad c_A \in \mathbb{C} \qquad \text{and} \qquad z_j = e^{i\xi_j}. \tag{2.25}$$

In [19] we studied polynomials orthogonal with respect to the more general linear functional

$$\mathscr{L}(h; \mathscr{A}, \mathscr{W}, \lambda) := \frac{1}{2\pi}\int_{E_l} h(e^{i\varphi})\, f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi + \mathscr{G}(h; \mathscr{A}, \mathscr{W}, \lambda) \tag{2.26}$$

with, recalling the notation in (2.20), resp. (2.25),

$$\begin{aligned} \mathscr{G}(h; \mathscr{A}, \mathscr{W}, \lambda) &:= -\frac{1}{2}\sum_{j=1}^{m^*} (1-\lambda_j) \sum_{v=0}^{m_j-1} \mu_{j,v}(-1)^v\, \delta_{z_j}^{(v)}\left(\frac{h(z)}{z}\right) \\ &= -\frac{1}{2}\sum_{j=1}^{m^*} \frac{1-\lambda_j}{(m_j-1)!}\left(\frac{z^{a+l/2-w-1}Wh}{A_j\sqrt{R}}\right)^{(m_j-1)}(z_j), \end{aligned} \tag{2.27}$$

where $\lambda_j \in \{-1, +1\}$, i.e., if $\lambda_j = -1$ then there appears a "Dirac mass-point" at z_j, where $\delta_{z_j}^{(v)}$ is defined by $\delta_{z_j}^{(v)}(g) := (-1)^v g^{(v)}(z_j)/v!$ and where the constants $\mu_{j,v}$ are given by

$$\mu_{j,v} := \frac{1}{(m_j-1-v)!}\left(\frac{z^{a+l/2-w}W}{A_j\sqrt{R}}\right)^{(m_j-1-v)}(z_j), \qquad A_j(z) := \frac{A(z)}{(z-z_j)^{m_j}}.$$

To be more precise, $\lambda = \{\lambda_1, ..., \lambda_{m^*}\} \in \Delta_{m^*}$, where

$$\Delta_{m^*} := \{(\lambda_1, ..., \lambda_{m^*})\colon \lambda_j \in \{-1, +1\} \text{ and } \lambda_{j_1} = \lambda_{j_2} \text{ if } z_{j_1} = 1/\overline{z_{j_2}}\}. \tag{2.28}$$

Hence we described those polynomials P_n of degree n which satisfy the orthogonality condition

$$\mathscr{L}(x^{-k}P_n; \mathscr{A}, \mathscr{W}, \lambda) = \frac{1}{2\pi}\int_{E_l} e^{-ik\varphi} P_n(e^{i\varphi}) f(\varphi; \mathscr{A}, \mathscr{W})\, d\varphi - \frac{1}{2}\sum_{j=1}^{m^*} (1-\lambda_j) \sum_{\nu=0}^{m_j-1} \mu_{j,\nu}(-1)^\nu\, \delta_{z_j}^{(\nu)}(x^{-(k+1)}P_n) = 0$$

$$\text{for} \quad k = 0, ..., n-1. \tag{2.29}$$

Examples of what kind of "weight functions" and orthogonality measures are covered by the functionals $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ are given in [19].

3. COMPLEX T-POLYNOMIALS

We start with the essential

DEFINITION 3.1. Let E_l and thus R be given by (2.15) and (2.23), respectively. Let $N \geqslant l$, $N \in \mathbb{N}$, and let $\mathscr{T}_N^*$ and $\mathscr{U}_{N-l}^*$ be selfreciprocal polynomials of degree N and $N-l$, respectively, where the leading coefficient α of $\mathscr{T}_N$ is normalized by $|\alpha| = 1$. If $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ satisfy

$$\mathscr{T}_N^2(z) - R(z)\, \mathscr{U}_{N-l}^2(z) = L^2 z^N, \qquad L \in \mathbb{R}^+, \tag{3.1}$$

where the sign of $\mathscr{U}_{N-l}$ is chosen such that $\mathscr{T}_N(0) = \sqrt{R(0)}\, \mathscr{U}_{N-l}(0)$, then $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ are called complex Chebyshev polynomials, abbreviated by complex T-polynomials, on the set E_l of the first and second kind, respectively.

Note that because of the one-to-one relation between the polynomial R and the set E_l, the existence of a complex T-polynomial only depends on the set E_l. Note further that by the selfreciprocity of $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$,

$$\tau_N(\varphi) := e^{-i(N/2)\varphi}\mathscr{T}_N(e^{i\varphi}), \qquad u_{N-l}(\varphi) := e^{-i((N-l)/2)\varphi}\mathscr{U}_{N-l}(e^{i\varphi}) \tag{3.2}$$

are real trigonometric polynomials of degree $N/2$ and $(N-l)/2$, respectively, which satisfy by (3.1)

$$\tau_N^2(\varphi) - \mathscr{R}(\varphi)\, u_{N-l}^2(\varphi) = L^2. \tag{3.3}$$

The reason why we call the polynomials $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ complex Chebyshev polynomials is given a little bit later in Corollary 3.2. Real T-polynomials have been investigated by the first author in [15, Section 2; 16].

Remark 3.1. From Definition 3.1 there immediately follows: If $\mathcal{T}_N(z) = \alpha z^N + \cdots$ is a complex T-polynomial on the set $E_l = \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}]$ then

$$\hat{\mathcal{T}}_N(z) := d^{N/2} \mathcal{T}_n(\tfrac{z}{d}) = z^N + \cdots \qquad \text{where} \quad d = \alpha^{2/N} \tag{3.4}$$

is a monic complex T-polynomial on the set $\hat{E}_l = \bigcup_{j=1}^{l} [\varphi_{2j-1} + \arg d, \varphi_{2j} + \arg d] =: E_l + \arg d$. The corresponding equation (3.1) is of the form

$$\hat{\mathcal{T}}_N^2(z) - \hat{R}(z)\, \hat{\mathcal{U}}_{N-l}^2(z) = L^2 z^N \qquad \text{(the same } L \text{ as in (3.1))}, \tag{3.5}$$

where $\hat{R}(z) := d^l R(z/d)$ and $\hat{\mathcal{U}}_{N-l}(z) := d^{(N-l)/2} \mathcal{U}_{N-l}(z/d)$. On the other hand, if there exists a monic complex T-polynomial on E_l, then for all $\delta \in \mathbb{R}$ there exists a complex T-polynomial on $E_l - \delta$ with leading coefficient $\alpha = e^{i(N\delta/2)}$ and again these complex T-polynomials on E_l and $E_l - \delta$ are related to each other by an equation of the form (3.4) with $d = e^{i\delta}$.

Let us now consider some simple examples.

Example 3.1. (a) On every interval $[\tilde{\varphi}_1, \tilde{\varphi}_2]$, $\tilde{\varphi}_1 < \tilde{\varphi}_2$ and $\tilde{\varphi}_2 - \tilde{\varphi}_1 \leqslant 2\pi$, there exists a complex T-polynomial of degree one. Indeed, let us first consider $E_1 = [\varphi_1, 2\pi - \varphi_1]$, $0 \leqslant \varphi_1 < \pi$. Then $R(z) = z^2 - 2z \cos \varphi_1 + 1$ and, in view of (3.1),

$$\mathcal{T}_1(z) := z + 1 \qquad \text{where} \quad L^2 := 2 + 2\cos \varphi_1 > 0$$

is a complex T-polynomial on E_1. Now as a consequence of Remark 3.1, $\tilde{\mathcal{T}}_1(z) := e^{i(\delta/2)} z + e^{-i(\delta/2)}$, $\delta \in \mathbb{R}$, is a complex T-polynomial on the set $\tilde{E}_1 = [\tilde{\varphi}_1, \tilde{\varphi}_2]$, where $\tilde{\varphi}_1 := \varphi_1 - \delta$ and $\tilde{\varphi}_2 := 2\pi - \varphi_1 - \delta$.

(b) On the union of two intervals $E_2 = [\varphi_1, \varphi_2] \cup [\varphi_3, \varphi_4]$, $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$, $\varphi_4 - \varphi_1 < 2\pi$, there exists a complex T-polynomial of degree two if and only if $\varphi_2 - \varphi_1 = \varphi_4 - \varphi_3$.

Proof. Necessity. Let $\mathcal{T}_2$ be a complex T-polynomial on E_2. From (2.17) and (3.3) it immediately follows that $|\tau_2(\varphi)| < L$ on $\operatorname{Int}(E_2)$, $|\tau_2(\varphi)| > L$ outside E_2, and $|\tau_2(\varphi_j)| = L$, $j = 1, ..., 4$. Since τ_2—and also its derivative—is a trigonometric polynomial of degree 1, it can have at most two zeros on $[0, 2\pi)$; thus we conclude from the above properties of τ_2 that the zeros of τ_2 are simple, say σ_1 and σ_2, where $\sigma_1 \in (\varphi_1, \varphi_2)$ and $\sigma_2 \in (\varphi_3, \varphi_4)$. Thus the identity (3.3) can be written as

$$\begin{aligned} \mathcal{R}(\varphi) &= \tau_2^2(\varphi) - L^2 \\ &= \text{const} \cdot \sin^2\left(\frac{\varphi - \sigma_1}{2}\right) \sin^2\left(\frac{\varphi - \sigma_2}{2}\right) - L^2, \quad \text{const} \in \mathbb{R}^+. \end{aligned} \tag{3.6}$$

Let $\eta := (\sigma_1 + \sigma_2)/2$, then we have $\tau_2^2(\eta - \varphi) = \tau_2^2(\eta + \varphi)$, i.e., τ_2^2 is symmetric with respect to η. Since E_2 is exactly that set where $\mathscr{R}(\varphi) \leqslant 0$, this means that E_2 consists of two intervals of the same length.

Sufficiency. By comparing coefficients in (3.1) and by using the representation (2.23) of R, defined on the set $E_2 = [\varphi_1, \varphi_1 + \gamma] \cup [\varphi_3, \varphi_3 + \gamma]$, where $\gamma \in \mathbb{R}^+$ is such that $\varphi_1 + \gamma < \varphi_3$, $\varphi_3 + \gamma < \varphi_1 + 2\pi$, one gets that $\mathscr{T}_2(z) = \alpha z^2 + \beta z + \bar{\alpha}$, where $\alpha = e^{-(i/2)(\varphi_1 + \varphi_3 + \gamma)}$ and $\beta = -2\cos(\gamma/2)\cos(\varphi_1 - \varphi_3)/2$, is a complex T-polynomial on E_2.

For examples treating more general cases (e.g., $l > 2$ or $N > l$) the method of comparing coefficients in (3.1) in order to calculate complex T-polynomials becomes tedious. But sometimes it is possible to obtain complex T-polynomials in a simpler way (see Example 3.2 below).

(c) If $\mathscr{T}_N$ is a complex T-polynomial on E_l then by (3.1) $\mathscr{T}_{mN}(z) := \mathscr{T}_N(z^m)$, $m \in \mathbb{N}$, is a complex T-polynomial on $E_{ml} := \bigcup_{v=0}^{m-1} \bigcup_{j=1}^{l} [(\varphi_{2j-1} + 2v\pi)/m, (\varphi_{2j} + 2v\pi)/m]$, i.e., the set E_l "appears" m-times in E_{ml}.

(d) Complex T-polynomials for symmetric arcs can be obtained from the so-called real T-polynomials on several intervals investigated in [15]. Indeed, let $-1 < q_2 < q_3 < \cdots < q_{2l-1} < 1$ and let $C_N(x) = x^N + \cdots$ and $D_{N-l}(x) = x^{N-l} + \cdots$ be real polynomials (so-called real T-polynomials) such that for all $x \in [-1, 1]$,

$$C_N^2(x) + \prod_{j=2}^{2l-1} (x - q_j)(1 - x^2)\, D_{N-l}^2(x) = \text{const.} \tag{3.7}$$

Setting $x = \frac{1}{2}(z + z^{-1})$ and $\cos\varphi_j := q_{2l-j}$, $j = 1, ..., 2l-2$, we get

$$\begin{aligned}((2z)^N C_N(\tfrac{1}{2}(z + z^{-1})))^2 - \left(-\prod_{j=1}^{2l-2} (z - e^{i\varphi_j})(z - e^{-i\varphi_j}) \right) \\ \times (i(z^2 - 1)(2z)^{N-l} D_{N-l}(\tfrac{1}{2}(z + z^{-1})))^2 = \text{const} \cdot z^{2N},\end{aligned}$$

and thus the polynomials

$$\mathscr{T}_{2N}(x) := (2z)^N C_N(\tfrac{1}{2}(z + z^{-1})) \qquad \text{and}$$

$$\mathscr{U}_{2N-(2l-2)}(z) := i(z^2 - 1)(2z)^{N-l} D_{N-l}(\tfrac{1}{2}(z + z^{-1}))$$

are complex T-polynomials on the symmetric set of intervals

$$\begin{aligned}E_{2l-2} = [-\varphi_1, \varphi_1] \cup [\varphi_{2l-2}, 2\pi - \varphi_{2l-2}] \cup \bigcup_{j=1}^{l-2} ([\varphi_{2j}, \varphi_{2j+1}] \\ \cup [2\pi - \varphi_{2j+1}, 2\pi - \varphi_{2j}]).\end{aligned}$$

Let us point out that, as we have seen in the above exa
every set E_l of the form (2.15) does there exist a complex
But on the other hand, every selfreciprocal polynomial $\mathscr{T}$,
simple zeros lying on the unit circumference $|z|=1$, is a comple
on the set

$$E(\mathscr{T}, L) := \{\varphi \in [d, d+2\pi]: |\mathscr{T}(e^{i\varphi})| \leqslant L\}$$

$$\text{with} \quad L \leqslant \min\{|\mathscr{T}(e^{i\varphi})|: \tau'(\varphi) =$$

where $\tau(\varphi)$ is defined as in (3.2) by $\tau(\varphi) := e^{-i(\partial\mathscr{T}/2)\varphi}\mathscr{T}(e^{i\varphi}$
is chosen such that $|\mathscr{T}(e^{id})| \geqslant L$. If there are $(\partial\mathscr{T} -$
$j=1, ..., (\partial\mathscr{T}-l)$, such that $\tau'(\psi_j)=0$ and $|\mathscr{T}(ei\psi_j)|=L$,
$(\partial\mathscr{T}-l)$ "interior" extremal points of τ on $E(\mathscr{T}, L)$,
$E(\mathscr{T}, L) =: E_l$ consists of l disjoint intervals (see Fig. 1). In
selfreciprocal polynomial which vanishes exactly at th
$j=1, ..., (\partial\mathscr{T}-l)$, is the complex T-polynomial of the secon

By considering the construction of the set $E(\mathscr{T}, L)$ one
same polynomial $\mathscr{T}$ can be a complex T-polynomial on
$E(\mathscr{T}, L)$, because $E(\mathscr{T}, L)$ depends, as indicated by the no
the value of L.

Next let us show that the complex T-polynomials on a s
orthogonality properties on E_l. First we need

LEMMA 3.1. *Let $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ be complex T-polynomials*
second kind, respectively. Then there hold:

(a) $\mathscr{T}_N(0) \neq 0$ *and* $\mathscr{U}_{N-l}(0) \neq 0$.

(b) $\mathscr{T}_N$ *and* $\mathscr{U}_{N-l}$ *have no zero in common.*

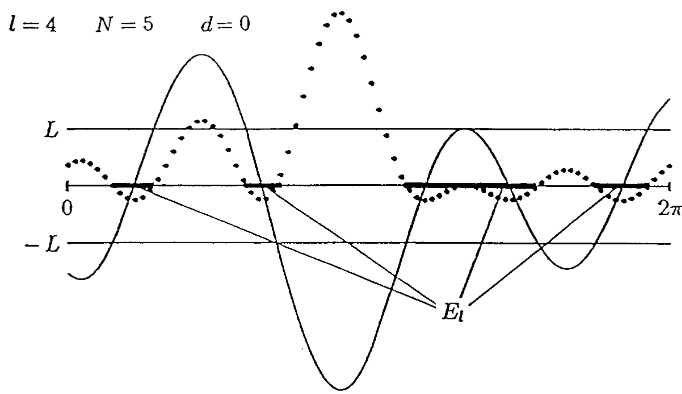

FIGURE 1.

(c) *Condition* (3.1) *is equivalent to the condition*

$$\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z) = \dot{O}(z^N), \qquad |z| < 1. \tag{3.8}$$

Proof. Part (a) is an immediate consequence of $\partial \mathcal{T}_N = N$, $\partial \mathcal{U}_{N-l} = N - l$, $\mathcal{T}_N = \mathcal{T}_N^*$, and $\mathcal{U}_{N-l} = \mathcal{U}_{N-l}^*$.

Suppose now that $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ have a common zero at z_0 with $z_0 \neq 0$ by part (a). Thus the left-hand side in (3.1) vanishes at z_0 while the right-hand side is unequal to zero. This is a contradiction and proves part (b).

(c) Necessity. From $\mathcal{T}_N(0) = \sqrt{R(0)}\, \mathcal{U}_{N-l}(0)$ and $\mathcal{T}_N(0) \neq 0$, i.e., $\mathcal{T}_N(0) + \sqrt{R(0)}\, \mathcal{U}_{N-l}(0) \neq 0$, we get by (3.1)

$$\begin{aligned}(\mathcal{T}_N(z) - \sqrt{R(z)}\, \mathcal{U}_{N-l}(z))(\mathcal{T}_N(z) + \sqrt{R(z)}\, \mathcal{U}_{N-l}(z)) \\ = \mathcal{T}_N^2(z) - R(z)\, \mathcal{U}_{N-l}^2(z) = \dot{O}(z^N)\end{aligned} \tag{3.9}$$

and (3.8) follows.

Sufficiency. By (3.9), which now holds because of (3.8), we can write

$$\mathcal{T}_N^2(z) - R(z)\, \mathcal{U}_{N-l}^2(z) = L_N z^N + \cdots + L_{2N} z^{2N},$$

$$L_N, ..., L_{2N} \in \mathbb{C}, \quad L_N \neq 0.$$

Since the polynomial at the left-hand side is selfreciprocal we get, by calculating the (modified) reciprocal polynomials at both sides,

$$\mathcal{T}_N^2(z) - R(z)\, \mathcal{U}_{N-l}^2(z) = L_N z^N \qquad \text{and} \qquad L_N \in \mathbb{R} \backslash \{0\}.$$

It remains to show that $L_N > 0$. In the same way as in (3.3) we get

$$\tau_N^2(\varphi) - \mathcal{R}(\varphi)\, u_{N-l}^2(\varphi) = L_N,$$

where the left-hand side is positive on $\operatorname{Int}(E_l) \neq \varnothing$ (note $\varphi_1 < \cdots < \varphi_{2l}$) since $\mathcal{R}(\varphi) < 0$ on $\operatorname{Int}(E_l)$. This gives the assertion. ∎

The following theorem is important in describing and characterizing complex T-polynomials.

THEOREM 3.1. *Let* $r(\varphi)$ *be given as in* (2.22) *and let* $\mathcal{T}_N$ *and* $\mathcal{U}_{N-l}$ *be complex T-polynomials on* E_l. *Then there hold*:

(a) $\mathcal{T}_N$ *and* $\mathcal{U}_{N-l}$ *are uniquely determined up to the factor* -1 *by Definition* 3.1.

(b) *$\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ are orthogonal with respect to $1/r(\varphi)$ and $r(\varphi)$, respectively; to be more precise,*

$$\int_{E_l} e^{-i(l/2+j)\varphi} \mathcal{T}_N(e^{i\varphi}) \frac{1}{r(\varphi)} \, d\varphi = 0 \qquad \text{for} \quad j = -l+1, ..., N-1, \tag{3.10}$$

$$\int_{E_l} e^{-i(l/2+j)\varphi} \mathcal{U}_{N-l}(e^{i\varphi}) \, r(\varphi) \, d\varphi = 0 \qquad \text{for} \quad j = -l+1, ..., N-l-1, \tag{3.11}$$

and the above orthogonality orders are exact.

(c) *All zeros of $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ are simple and lie on* $\mathrm{Int}(\Gamma_{E_l})$. *Further, $\mathcal{T}_N$ has at least one zero on each arc of Γ_{E_l}.*

Proof. We first give a proof for the case that l is even.

Add (b) and (a). The orthogonality properties follow by the same methods we used to prove Theorem 3.1 in [20] or by applying (the strong version of) Theorem 4.1 below, where we set $A(z) := iz^{l/2} \in \mathbb{P}_l$, $W := R$, $P_n = \mathcal{U}_{N-l}$, and use (3.8) to prove (3.11) and [19, Corollary 2.4] to prove (3.10).

Next let us demonstrate that the above orthogonality order of $\mathcal{T}_N$ is exact (the exact orthogonality order of $\mathcal{U}_{N-l}$ can be shown in the same way). Assuming the opposite, then by [20, Theorem 2.1(b)], noting that $\mathcal{T}_N(0) \neq 0$, there even holds

$$\int_{E_l} e^{-ij\varphi} \mathcal{T}_N(e^{i\varphi}) \frac{d\varphi}{r(\varphi)} = 0 \qquad \text{for} \quad j = -\frac{l}{2}, ..., N + \frac{l}{2}.$$

But this implies that $\int_{E_l} |\mathcal{T}_N(e^{i\varphi})|^2 \, (t(\varphi)/r(\varphi)) \, d\varphi = 0$ for each trigonometric polynomial $t(\varphi)$ of degree $l/2$, which is not possible by choosing $t(\varphi)$ such that $t(\varphi)/r(\varphi) > 0$ on $\mathrm{Int}(E_l)$. Hence part (b) is proved. Now by Lemma 3.1(a) and [20, Proposition 2.2] $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ are, in the notation of [20], so-called basic polynomials, which are uniquely determined up to the factor -1 (note $|\alpha| = 1$). This proves part (a) of the theorem.

Add (c). By the uniqueness and the high orthogonality property of $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ there follows from [2, Theorem 8] that all the zeros of $\mathcal{T}_N$ and $\mathcal{U}_{N-l}$ are simple and have unit modulus. Further, by (2.17) and (3.3) we have $|\mathcal{T}_N(e^{i\varphi})| = |\tau_N(\varphi)| \geq L$ for φ outside E_l, hence all the zeros of τ_N are in $\mathrm{Int}(E_l)$. The fact that τ_N has at least one zero on each interval of E_l holds, because otherwise τ_N, resp. τ'_N, would have too many zeros on an interval of the length 2π.

In addition to the above shown properties it even holds that $|\tau_N(\varphi)| > L$ outside E_l, because again in the opposite case τ'_N would have too many zeros. From (3.3) we see that u_{N-l} can only vanish at points where

$\tau_N^2(\varphi) = L^2$. Now the facts that $|\tau_N(\varphi)| > L$ outside E_l and $\tau_N'(\varphi_j) \neq 0$, $j = 1, ..., 2l$, together with the differentiated equation in (3.3), yield that all the zeros of u_{N-l} are in $\mathrm{Int}(E_l)$ as well. By (3.2) this proves the theorem for the case that l is even.

For the case that l is odd we consider the polynomials

$$\tilde{R}(z) := R(z^2), \qquad \tilde{\mathscr{T}}_{2N}(z) := \mathscr{T}_N(z^2), \qquad \tilde{\mathscr{U}}_{2N-2l}(z) := \mathscr{U}_{N-l}(z^2).$$

Then $\tilde{\mathscr{T}}_{2N}$ and $\tilde{\mathscr{U}}_{2N-2l}$ are complex T-polynomials on $E_{2l} := \{\varphi : e^{-2il\varphi} R(e^{2i\varphi}) \leqslant 0\}$, which consists of $2l$ intervals; compare with Example 3.1(c). Now $2l$ is even and all assertions can be obtained from the already proven results for even l's. ▮

From Theorem 3.1 a characterization of complex T-polynomials can be derived. First we need the following notation: Let $(c_k)_{k \in \mathbb{Z}}$ be a given double-sided infinite sequence of complex numbers satisfying $c_{-k} = \overline{c_k}$. Then the value $\Delta_j^{(n)}(\{c_k\})$ is defined by

$$\Delta_j^{(n)}(\{c_k\}) := \det \begin{pmatrix} c_j & c_{j-1} & \cdots & c_{j-n} \\ c_{j+1} & c_j & \cdots & c_{j+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{j+n} & c_{j+n-1} & \cdots & c_j \end{pmatrix}, \quad n \geqslant 0,$$

$$\text{and} \qquad \Delta_j^{(-1)}(\{c_k\}) := 1.$$

Corollary 3.1. *There exists a complex T-polynomial $\mathscr{T}_N$ on a set E_l if and only if there holds the following condition* (3.12) *if l is even and* (3.13) *if l is odd.*

$$\Delta_0^{(N-l-1)}(\{c_k\}) \neq 0 \qquad \textit{and} \qquad \Delta_j^{(N-l)}(\{c_k\}) = 0$$
$$\textit{for} \quad j = -\frac{l}{2} + 1, ..., \frac{l}{2} - 1, \tag{3.12}$$

where $c_k := \int_{E_l} e^{-ik\varphi} r(\varphi)\, d\varphi$.

$$\Delta_0^{(2(N-l)-1)}(\{\tilde{c}_k\}) \neq 0 \qquad \textit{and} \qquad \Delta_j^{(2(N-l))}(\{\tilde{c}_k\}) = 0$$
$$\textit{for} \quad j = -l + 1, ..., l - 1, \tag{3.13}$$

where $\tilde{c}_k := \int_{E_l} e^{-i(k/2)\varphi} r(\varphi)\, d\varphi$ *if k is odd and* $\tilde{c}_k := 0$ *if k is even.*

Proof. We first consider the case that l is even.

Necessity. Let $\mathcal{U}_{N-l}(z) = d_{N-l} z^{N-l} + \cdots + d_0$ be the complex T-polynomial of the second kind on E_l. Writing down the orthogonality property (3.11) as a system of equations in terms of the moments (c_k), i.e.,

$$\sum_{m=0}^{N-l} c_{j-m} d_m = 0 \qquad \text{for} \quad j = -\frac{l}{2}+1, ..., N-\frac{l}{2}-1,$$

and using the uniqueness of $\mathcal{U}_{N-l}$, i.e., $\Delta_0^{(N-l-1)}(\{c_k\}) \neq 0$, then (3.12) follows.

Sufficiency. From the system (3.12) one can derive by standard methods that the rank of the matrix $(\mathbf{c}_{-l/2+1}, ..., \mathbf{c}_{N-l/2-1})^T$ is $N-l$, where $\mathbf{c}_j := (c_j, c_{j-1}, ..., c_{j-(N-l)})$. Hence, there exists a polynomial $\tilde{\mathcal{U}}_{N-l}$ of degree $N-l$ such that

$$\int_{E_l} e^{-ij\varphi} \tilde{\mathcal{U}}_{N-l}(e^{i\varphi})\, r(\varphi)\, d\varphi = \sum_{m=0}^{N-l} c_{j-m} \tilde{d}_m = 0$$

$$\text{for} \quad j = -\frac{l}{2}+1, ..., N-\frac{l}{2}-1,$$

where the $\tilde{d}_m$'s denote the coefficients of $\tilde{\mathcal{U}}_{N-l}$. As shown in the proof of Theorem 3.1, by this high orthogonality property $\tilde{\mathcal{U}}_{N-l}$ is, in the terminology of [20], a basic polynomial and thus it satisfies $\tilde{\mathcal{U}}^*_{N-l} = \mu \tilde{\mathcal{U}}_{N-l}$ with $|\mu| = 1$ (see e.g. [20, Proposition 2.2(a)]).

Now we can set $A(z) := iz^{l/2}$, $W := R$, and $P_n := \tilde{\mathcal{U}}_{N-l}$, and get by (the strong version of) Theorem 4.1 below that $\tilde{\mathcal{T}}_N(z) - \sqrt{R(z)}\, \tilde{\mathcal{U}}_{N-l}(z) = \dot{O}(z^N)$, where $\tilde{\mathcal{T}}_N$ is a selfreciprocal polynomial of degree N which satisfies $\tilde{\mathcal{T}}^*_N = \mu \tilde{\mathcal{T}}_N$(compare [19, (3.33)] and [20, Proposition 2.2(a)]). Now the normalization $\mathcal{T}_N := s\mu^{1/2} \tilde{\mathcal{T}}_N$, where $s \in \mathbb{R} \backslash \{0\}$ is chosen such that the leading coefficient of $\mathcal{T}_N$ has modulus 1, and Lemma 3.1(c) proves the theorem for the case that l is even.

For the case that l is odd we consider, as in the proof of Theorem 3.1, the polynomial $\tilde{R}(z) := R(z^2)$ of degree $4l$. Let $\tilde{E}_{2l}$ and $\tilde{r}(\varphi)$ be the corresponding set and function from (2.15) and (2.22), respectively. Now by the already proven part of the theorem we get the characterization (3.13), where the moments $\tilde{c}_k$ are given by $\tilde{c}_k = \int_{\tilde{E}_{2l}} e^{-ik\varphi} \tilde{r}(\varphi)\, d\varphi$. Using the fact that $\tilde{r}(\varphi) = -\tilde{r}(\varphi + \pi)$ and $\tilde{r}(\varphi) = r(2\varphi)$, $\varphi \in [0, \pi]$, the moments $\tilde{c}_k$ can also be written as in the theorem. ∎

With the help of Corollary 3.1 we now give a characterization of the existence of a complex T-polynomial $\mathcal{T}_3$ on three and on two arcs (see also Example 3.1).

EXAMPLE 3.2. (a) Complex T-polynomial $\mathscr{T}_3$ on three arcs: Let $E_3=[\varphi_1,\varphi_2]\cup[\varphi_3,\varphi_4]\cup[\varphi_5,\varphi_6]$ be an arbitrary set of the form (2.15) and let $R_6(z)=\varrho_0+\varrho_1 z+\varrho_2 z^2+\varrho_3 z^3+\bar{\varrho}_2 z^4+\bar{\varrho}_1 z^5+\bar{\varrho}_0 z^6$ be the corresponding selfreciprocal polynomial from (2.23). Then there exists a complex T-polynomial $\mathscr{T}_3$ on E_3 if and only if

$$\varrho_1^2-4\varrho_2\varrho_0+4\bar{\varrho}_1\varrho_0^2=0.$$

(b) Complex T-polynomial $\mathscr{T}_3$ on two arcs: Let the set $E_2=[\varphi_1,\varphi_2]\cup[\varphi_3,\varphi_4]$ and the polynomial $R_4(z)=\varrho_0+\varrho_1 z+\varrho_2 z^2+\bar{\varrho}_1 z^3+\bar{\varrho}_0 z^4$ be given as in (a). Then there exist complex T-polynomials $\mathscr{T}_3$ and $\mathscr{U}_1$ on E_2 if and only if

$$\operatorname{Im}\left\{\frac{\varrho_1}{\sqrt{\varrho_0}}\right\}\neq 0 \quad \text{and} \quad (4\varrho_2-8)^2+(|\varrho_1|^2-16)^2+8\operatorname{Re}\{\bar{\varrho}_0\varrho_1^2\}(6-\varrho_2)=2^8.$$

Proof. (a) By (3.13) there exists a complex T-polynomial $\mathscr{T}_3$ on E_3 if and only if $\tilde{c}_1=0$ where $\tilde{c}_1=\int_{E_3} e^{-i(\varphi/2)}r(\varphi)\,d\varphi$ (note $\tilde{c}_{-1}=\overline{\tilde{c}_1}$). By [20, Theorem 2.2] (compare also [20, Theorem 3.1]) we get

$$\frac{(1/i)\sqrt{R_6(z^2)}-\Phi_6(z)}{z^3}=\frac{1}{2\pi}\int_{\tilde{E}_6}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\tilde{r}(\varphi)\,d\varphi=\tilde{c}_0+2\sum_{k=1}^{\infty}\tilde{c}_k z^k,$$

where $\Phi_6\in\mathbb{P}_6$ with $\Phi_6^*=-\Phi_6$. To be more precise, if we write

$$\frac{1}{i}\sqrt{R_6(z^2)}=:d_0+d_2z^2+d_4z^4+\cdots, \tag{3.14}$$

which is possible since R_6 is analytic in $|z|<1$ and has no zeros there, then

$$\Phi_6(z)=d_0+d_2z^2-\bar{d}_2z^4-\bar{d}_0z^6.$$

Hence, we get the representation

$$\begin{aligned}\frac{(1/i)\sqrt{R_6(z^2)}-\Phi_6(z)}{z^3}&=(d_4+\bar{d}_2)\,z+(d_6+\bar{d}_0)\,z^3+d_8z^5+\cdots\\&=\tilde{c}_1z+2\tilde{c}_3z^3+2\tilde{c}_5z^5+\cdots.\end{aligned}$$

By calculating the first three coefficients of the series expansion of $\sqrt{R_6(z^2)}$ in terms of the coefficients ϱ_j of R_6, there follows $d_0=i\sqrt{\varrho_0}$, $d_2=i\varrho_1/2\sqrt{\varrho_0}$, $d_4=\varrho_1^2-4\varrho_2\varrho_0/8i\varrho_0\sqrt{\varrho_0}$, and thus

$$\tilde{c}_1=d_4+\bar{d}_2=\frac{\varrho_1^2\sqrt{\bar{\varrho}_0}-4\varrho_2\varrho_0\sqrt{\bar{\varrho}_0}+4\bar{\varrho}_1\varrho_0\sqrt{\varrho_0}}{8i\varrho_0}.$$

Taking into consideration the fact that $\varrho_0 = R_6(0)$ and thus $|\varrho_0| = 1$ (note (2.23)), the assertion follows.

(b) By (3.12) and $c_{-1} = \bar{c}_1$ the existence of the complex T-polynomials $\mathscr{T}_3$ and $\mathscr{U}_1$ on E_2 is equivalent to

$$c_0 \neq 0 \qquad \text{and} \qquad c_0^2 - |c_1|^2 = 0. \tag{3.15}$$

Similarly to (a) we have now

$$\frac{(1/i)\sqrt{R_4(z)} - \Phi_2(z)}{z} = \frac{1}{2\pi}\int_{E_2} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, r(\varphi)\, d\varphi = c_0 + 2\sum_{k=1}^{\infty} c_k z^k,$$

where $\Phi_2 \in \mathbb{P}_2$ with $\Phi_2^* = -\Phi_2$, and again we write $(1/i)\sqrt{R_4(z)} =: d_0 + d_1 z + d_2 z^2 + \cdots$ and get explicitly $\Phi_2(z) = d_0 + i\,\mathrm{Im}\{d_1\} z - \bar{d}_0 z^2$. Using the representation of R_4 one easily calculates $d_0 = i\sqrt{\varrho_0}$, $d_1 = i\varrho_1/2 \times \sqrt{\varrho_0}$, $d_2 = (i\sqrt{\varrho_0}/2)(\varrho_2/\varrho_0 - \varrho_1^2/4\varrho_0^2)$ and thus

$$c_0 = \frac{1}{2}\,\mathrm{Re}\left\{\frac{i\varrho_1}{\sqrt{\varrho_0}}\right\}, \qquad c_1 = \frac{8\varrho_0 + \varrho_1^2 - 4\varrho_0\varrho_2}{16 i \varrho_0 \sqrt{\varrho_0}}.$$

Substituting these expressions in (3.15) gives after some straightforward calculation the assertion.

From Theorem 3.1 one obtains the following corollary, which gives the reason why we call the polynomials $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ complex Chebyshev polynomials.

COROLLARY 3.2. *Let $\mathscr{T}_N$ and $\mathscr{U}_{N-l}$ be complex T-polynomials on E_l with leading coefficients $\alpha =: e^{-i\psi}$ and $\beta = \sqrt{R(0)}\,\alpha =: e^{-i\eta}$, respectively, and let $\tau_N(\varphi) = e^{-i(N/2)\varphi}\mathscr{T}_N(e^{i\varphi})$ and $u_{N-l}(\varphi) = e^{-i((N-l)/2)\varphi}\mathscr{U}_{N-l}(e^{i\varphi})$. Then the following statements hold.*

(a) *The trigonometric polynomial $\frac{1}{2}\tau_N$ deviates least from zero on E_l with respect to the sup-norm among all trigonometric polynomials of degree $N/2$ with leading coefficients* $\cos\psi$ *and* $\sin\psi$, *i.e.,*

$$\max_{\varphi\in E_l}\left|\frac{1}{2}\tau_N(\varphi)\right| = \inf_{c_j, d_j \in \mathbb{R}} \max_{\varphi \in E_l}\left|\cos\psi\cos\frac{N}{2}\varphi + \sin\psi\sin\frac{N}{2}\varphi + \sum_{j=1}^{\lfloor N/2 \rfloor} c_j \cos\frac{N-2j}{2}\varphi + d_j \sin\frac{N-2j}{2}\varphi\right|. \tag{3.16}$$

(b) *$\frac{1}{2}u_{N-l}$ deviates least from zero on E_l among all trigonometric polynomials of degree $(N-l)/2$ with leading coefficients* $\cos\eta$ *and* $\sin\eta$ *with* *respect to the sup-norm with weight function* $\sqrt{|\mathscr{R}(\varphi)|}$, $\mathscr{R}$ *defined in* (2.16), *i.e.*,

$$\max_{\varphi \in E_l} \left| \frac{1}{2} u_{N-l}(\varphi) \sqrt{\mathscr{R}(\varphi)} \right|$$

$$= \inf_{c_j, d_j \in \mathbb{R}} \max_{\varphi \in E_l} \left| \left(\cos \eta \cos \frac{N-l}{2} \varphi + \sin \eta \sin \frac{N-l}{2} \varphi \right. \right.$$

$$\left. \left. + \sum_{j=1}^{\lfloor (N-l)/2 \rfloor} c_j \cos \frac{N-l-2j}{2} \varphi + d_j \sin \frac{N-l-2j}{2} \varphi \right) \sqrt{\mathscr{R}(\varphi)} \right|. \quad (3.17)$$

(c) *Let* t_N *be a minimal trigonometric polynomial of degree* $N/2$ *on a set* E_l *of the form* (2.15) *in the sense of* (3.16) *and suppose that* t_N *has* $N+l$ *extremal points on* E_l. *Then* $z^{N/2} t_N(z)$ *is* (*up to a real multiplicative factor*) *a complex T-polynomial on* E_l.

Proof. (a) By (3.3) and (2.17) we have that $|\tau_N(\varphi)| \leqslant L$ on E_l. Further, we see that $|\tau_N(\varphi_j)| = |\tau_N(\psi_k)| = L$ at the boundary points $\varphi_1, ..., \varphi_{2l}$ of E_l and at the zeros $\psi_1, ..., \psi_{N-l}$ of u_{N-l}, which are all simple and are lying in $\mathrm{Int}(E_l)$ as we have shown in Theorem 3.1(c). Thus τ_N has (at least) $N+l$ extremal points on E_l. If we cancel out from these $N+l$ extremal points the boundary points φ_{2j+1}, $j=1, ..., l-1$, recall that $\tau_N(\varphi_{2j}) = \tau_N(\varphi_{2j+1})$, then the remaining points are $N+1$ alternating points of τ_N on E_l, since otherwise the trigonometric polynomial τ'_N would have too many zeros on an interval of length 2π. Now (3.16) follows from the well-known Alternation Theorem.

(b) We now consider the system of functions $\{\sqrt{|\mathscr{R}(\varphi)|} \cdot \cos((N-l-2j)/2)\, \varphi$, $\sqrt{|\mathscr{R}(\varphi)|} \cdot \sin((N-l-2j)/2)\, \varphi \colon j=0, ..., \lfloor (N-l)/2 \rfloor\}$, which is a Haar system on each set $E_l^{(\varepsilon)} := \{\varphi \in E_l \colon |\varphi - \varphi_j| \geqslant \varepsilon, j=1, ..., 2l\}$, where $\varepsilon > 0$. Again by (3.3) there hold $|\sqrt{\mathscr{R}(\varphi)}\, u_{N-l}(\varphi)| \leqslant L$ on E_l and $|\sqrt{\mathscr{R}(\sigma_j)}\, u_{N-l}(\sigma_j)| = L$ at the N distinct zeros $\sigma_1, ..., \sigma_N$ of τ_N which are contained in $\mathrm{Int}(E_l)$ and where there is at least one σ_j in each interval of E_l by Theorem 3.1(c). If one considers (the proof of) Theorem 3.1 in detail, one can see that between two zeros of τ_N, which are lying in the same interval, there is exactly one zero of u_{N-l}. Thus, if we cancel the smallest zero σ_j in each interval, then it is not very hard to show that the remaining $N-l$ zeros of τ_N are alternating points of u_{N-l} on $E_l^{(\varepsilon)}$, where ε is small enough that $E_l^{(\varepsilon)}$ contains all the zeros of τ_N. This means that again by the Alternation Theorem u_{N-l} is the minimal trigonometric polynomial with respect to the sup-norm with weight function $\sqrt{|\mathscr{R}(\varphi)|}$ on each of those sets $E_l^{(\varepsilon)}$. The assertion follows now by the limit process $\varepsilon \to 0$.

(c) Since t'_N is a trigonometric polynomial of degree $N/2$, it can have at most N zeros on each interval of the length 2π. Thus by carefully counting the zeros of t'_N one gets that all the boundary points of E_l must be extremal points of t_N. Thus the set of extremal points consists exactly of $\varphi_1, ..., \varphi_{2l}, \psi_1, ..., \psi_{N-l}$, where $\psi_j \in \mathrm{Int}(E_l)$ for $j=1, ..., N-l$. Let us denote the value of the sup-norm of $\frac{1}{2}t_N$ by L. Then by $|t_N(\varphi_j)| = |t_N(\psi_k)| = L$ and $t'_N(\psi_k) = 0$, $j=1, ..., 2l$ and $k=1, ..., N-l$, we have

$$t_N^2(\varphi) - L^2 =: \mathscr{R}(\varphi)\, \tilde{u}_{N-l}^2(\varphi),$$

where $\tilde{u}_{N-l}$ is a real trigonometric polynomial of degree $(N-l)/2$, which vanishes exactly at the points $\psi_1, ..., \psi_{N-l}$. Hence, the algebraic polynomials $z^{N/2} t_N(z)$ and $z^{(N-l)/2} \tilde{u}_{N-l}(z)$ are (up to a real multiplicative factor) complex T-polynomials on E_l. ∎

From (the proof of) Corollary 3.2 we get, in addition to Corollary 3.1, the following characterization of a complex T-polynomial on a set E_l: An algebraic selfreciprocal polynomial $\mathscr{T}_N^*$ with leading coefficient α, $|\alpha| = 1$, is a complex T-polynomial on E_l if and only if the trigonometric polynomial $\tau_N(\varphi) := e^{-i(N/2)\varphi} \mathscr{T}_N(e^{i\varphi})$ has exactly $N+l$ extremal points on E_l, where all the boundary points of E_l are among these extremal points.

We now demonstrate the fact that as soon as we know one complex T-polynomial on a set E_l we know all complex T-polynomials on E_l. In what follows let $T_n(x) = \cos(n \arccos x) = 2^{n-1}x^n + \cdots$ and $U_{n-1}(x) = \sin(n \arccos x)/(\sin \arccos x) = 2^{n-1}x^{n-1} + \cdots$, $n \in \mathbb{N}_0$, be the classical Chebyshev polynomials of the first and second kind on the interval $[-1, +1]$, respectively.

THEOREM 3.2. *Let $N \geqslant l$ be the smallest integer such that there exists a complex T-polynomial $\mathscr{T}_N(z) = \alpha z^N + \cdots$, $|\alpha| = 1$, on E_l and denote by $\mathscr{U}_{N-l}(z) = \beta z^{N-l} + \cdots$, $\beta = \sqrt{R(0)}\,\alpha$, the corresponding complex T-polynomial of the second kind. Then for all $n \in \mathbb{N}$*

$$\mathscr{T}_{nN}(z) := \frac{1}{2^{n-1}} (Lz^{N/2})^n\, T_n\left(\frac{\mathscr{T}_N(z)}{Lz^{N/2}}\right) = \alpha^n z^{nN} + \cdots \tag{3.18}$$

$$\mathscr{U}_{nN-l}(z) := \frac{1}{2^{n-1}} \mathscr{U}_{N-l}(z)(Lz^{N/2})^{n-1}\, U_{n-1}\left(\frac{\mathscr{T}_N(z)}{Lz^{N/2}}\right) = \alpha^{n-1}\beta z^{nN-l} + \cdots, \tag{3.19}$$

where L is given as in (3.1), *are complex T-polynomials on E_l, and besides the polynomials in* (3.18), (3.19) *there exists no other complex T-polynomial on E_l.*

Proof. The fact that $\mathcal{T}_{nN}$ and $\mathcal{U}_{nN-l}$, given in (3.18) and (3.19), are complex T-polynomials on E_l can be seen in quite the same way as we proved Lemma 3.1 in [20]. Now we have to show that there are no more complex T-polynomials on E_l, i.e., by Theorem 3.1(a) we have to show that there exists no complex T-polynomial $\mathcal{T}_\nu$ of degree $\nu \neq nN$. Assume the opposite, i.e., suppose that there exists a complex T-polynomial $\mathcal{T}_\nu$ of degree ν with $(n-1)\,N < \nu < nN$ for an arbitrary $n \in \mathbb{N} \backslash \{1\}$. By Definition 3.1 and (3.8) we have

$$\mathcal{T}_\nu^2(z) - R(z)\,\mathcal{U}_{\nu-l}^2(z) = L_\nu^2 z^\nu, \qquad \mathcal{T}_\nu(z) - \sqrt{R(z)}\,\mathcal{U}_{\nu-l}(z) = \dot{O}(z^\nu)$$

and

$$\mathcal{T}_{nN}^2(z) - R(z)\,\mathcal{U}_{nN-l}^2(z) = L_{nN}^2 z^{nN}, \qquad \mathcal{T}_{nN}(z) - \sqrt{R(z)}\,\mathcal{U}_{nN-l}(z) = \dot{O}(z^{nN}).$$

Multiplying the above quadratic equations we get

$$\begin{aligned}(L_\nu L_{nN})^2\, z^{\nu+nN} &= (\mathcal{T}_\nu^2 - R\mathcal{U}_{\nu-l}^2)(\mathcal{T}_{nN}^2 - R\mathcal{U}_{nN-l}^2) \\ &= (\mathcal{T}_\nu \mathcal{T}_{nN} - R\mathcal{U}_{\nu-l}\mathcal{U}_{nN-l})^2 - R(\mathcal{T}_\nu \mathcal{U}_{nN-l} - \mathcal{T}_{nN}\mathcal{U}_{\nu-l})^2.\end{aligned} \tag{3.20}$$

Substituting $\mathcal{T}_\nu = \sqrt{R}\,\mathcal{U}_{\nu-l} + \dot{O}(z^\nu)$ and $\mathcal{T}_{nN} = \sqrt{R}\,\mathcal{U}_{nN-l} + \dot{O}(z^{nN})$ in the expression $\mathcal{T}_\nu \mathcal{U}_{nN-l} - \mathcal{T}_{nN}\mathcal{U}_{\nu-l}$ we obtain that $\mathcal{T}_\nu \mathcal{U}_{nN-l} - \mathcal{T}_{nN}\mathcal{U}_{\nu-l} = \dot{O}(z^\nu)$ and as a consequence by $\nu < nN$ and (3.20) there follows that $\mathcal{T}_\nu \mathcal{T}_{nN} - R\mathcal{U}_{\nu-l}\mathcal{U}_{nN-l} = \dot{O}(z^\nu)$. Thus we get $\partial[\mathcal{T}_\nu \mathcal{U}_{nN-l} - \mathcal{T}_{nN}\mathcal{U}_{\nu-l}] = nN - l$ and $\partial[\mathcal{T}_\nu \mathcal{T}_{nN} - R\mathcal{U}_{\nu-l}\mathcal{U}_{nN-l}] = nN$ since these polynomials are selfreciprocal. Let us now define

$$\mathcal{T}_{nN-\nu} := \frac{\mathcal{T}_\nu \mathcal{T}_{nN} - R\mathcal{U}_{\nu-l}\mathcal{U}_{nN-l}}{z^\nu} \in \mathbb{P}_{nN-\nu}$$

$$\mathcal{U}_{(nN-\nu)-l} := \frac{\mathcal{T}_\nu \mathcal{U}_{nN-l} - \mathcal{T}_{nN}\mathcal{U}_{\nu-l}}{z^\nu} \in \mathbb{P}_{nN-\nu-l},$$

then $\mathcal{T}_{nN-\nu}(0) \neq 0$, $\mathcal{U}_{(nN-\nu)-l}(0) \neq 0$, $\mathcal{T}^*_{nN-\nu} = \mathcal{T}_{nN-\nu}$, and $\mathcal{U}^*_{(nN-\nu)-l} = \mathcal{U}_{(nN-\nu)-l}$. Hence, by (3.20) $\mathcal{T}_{nN-\nu}$ is a complex T-polynomial on E_l of degree $nN-\nu < N$, which contradicts the assumption that $\mathcal{T}_N$ is that complex T-polynomial on E_l of smallest degree. ∎

4. ORTHOGONAL POLYNOMIALS WITH PERIODIC REFLECTION COEFFICIENTS

In this section we assume that $f(\varphi; \mathcal{A}, \mathcal{W})$ is integrable and that $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ is a definite functional, which is—in the simplest case—

fulfilled if for example $f(\varphi; \mathscr{A}, \mathscr{W})$ is a weight function and $\lambda = (1, 1, ..., 1)$.

Let us start with some preliminaries, which we have shown in [19]: First of all, there exists a uniquely determined selfreciprocal polynomial $B := B(\cdot; A, W, \lambda) \in \mathbb{P}_{2a}$, $\lambda = (\lambda_1, ..., \lambda_{m^*}) \in \Delta_{m^*}$, i.e., $B = B_{2a}^{(*)}$, which satisfies the interpolation condition, recall the notation in (2.25),

$$(VB)^{(\nu)}(z_j) = -\lambda_j (z^{a+l/2-w} \sqrt{R})^{(\nu)}(z_j), \qquad \nu = 0, ..., m_j - 1, \quad j = 1, ..., m^*, \tag{4.1}$$

and the "zero condition"

$$\left. \frac{B(z) + z^{a+l/2-w}(W(z)/\sqrt{R(z)})}{A(z)} \right|_{z=0} \in \mathbb{R}.$$

Further, in [19, Theorem 2.1] we have shown that the Stieltjes transform takes the form

$$\begin{aligned} \mathscr{L}\left(\frac{x+z}{x-z}; \mathscr{A}, \mathscr{W}, \lambda\right) &= \frac{B(z) + z^{a+l/2-w}(W(z)/\sqrt{R(z)})}{A(z)} \\ &=: F(z; A, W, \lambda), \qquad z \in \mathbb{C} \backslash (\Gamma_{E_l} \cup \{z_j : \lambda_j = -1\}). \end{aligned} \tag{4.2}$$

Thus if $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ is positive definite then F is the Carathéodory function.

Orthogonal polynomials with respect to a definite functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ can be characterized in the following way:

THEOREM 4.1 (Peherstorfer and Steinbauer [19]). *Let the polynomials $R, V, W, A, B(\cdot; A, W, \lambda) =: B$ and the definite functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ be given and let P_n be a polynomial of degree n.*

(a) *There holds $a + l/2 - w \geqslant 2v - l$ and if further $n \geqslant a + l/2 - w$ then the following two properties are equivalent:*

(a.i) *P_n is an orthogonal polynomial with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$, i.e., $\mathscr{L}(x^{-j} P_n; \mathscr{A}, \mathscr{W}, \lambda) = 0$ for $j = 0, ..., n-1$.*

(a.ii) *There exists a polynomial Q_{n+l-2v} of degree $n + l - 2v$ which satisfies together with P_n the following system*

$$\begin{aligned} &(VQ_{n+l-2v})^{(\nu)}(z_j) = \lambda_j (\sqrt{R}\, P_n)^{(\nu)}(z_j), \\ &\qquad\qquad \nu = 0, ..., m_j - 1, \quad j = 1, ..., m^* \\ &V(z)\, Q_{n+l-2v}(z) - \sqrt{R(z)}\, P_n(z) = O(z^{n-(a+l/2-w)}), \qquad \text{as} \quad z \to 0 \\ &V(z)\, Q^*_{n+l-2v}(z) - \sqrt{R(z)}\, P^*_n(z) = O(z^{n-(a+l/2-w)+1}), \qquad \text{as} \quad z \to 0. \end{aligned} \tag{4.3}$$

If there exists such a polynomial Q_{n+l-2v} *then it is of the form*

$$Q_{n+l-2v}(z) = -\frac{\Omega_n(z)\, A(z) + P_n(z)\, B(z)}{z^{a+l/2-w}}, \tag{4.4}$$

where Ω_n *denotes the polynomial of the second kind of* P_n *with respect to* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$.

(b) *If* $n \geqslant a + l/2 + v$ *and if* P_n *and* A *have at most a simple zero in common (this is no loss of generality as we have pointed out in* [19, *Remark* 2.3(a)]) *then we have the following equivalence*:

(b.i) $\mathscr{L}(x^{-j}P_n; \mathscr{A}, \mathscr{W}, \lambda) = 0$ *for* $j = 0, ..., n-1$ *and* $\mathscr{L}(x^{\tau}P_n; \mathscr{A}, \mathscr{W}, \lambda) \neq 0$ *for at least one* $\tau \in \mathbb{N}$.

(b.ii) *There exists a polynomial* $Q_{n+l-2v} \in \mathbb{P}_{n+l-2v}$ *(and if this polynomial exists then it is of the form* (4.4)) *and a polynomial* $g_{(n)} \in \mathbb{P}_{l-1-p}$ *with* $g_{(n)}(0) \neq 0$, *where* $p \in \mathbb{N}_0$ *gives the order of the zero of* P_n *at* $z = 0$, *such that*

$$W(z)\, P_n^2(z) - V(z)\, Q_{n+l-2v}^2(z) = z^{n+p-(a+l/2-w)} A(z)\, g_{(n)}(z) \tag{4.5}$$

and ("*sign-condition*")

$$V(z_j)\, Q_{n+l-2v}(z_j) = \lambda_j \sqrt{R(z_j)}\, P_n(z_j), \qquad j = 1, ..., m^* \tag{4.6}$$

$$\left.\frac{VQ_{n+l-2v}}{\sqrt{R}\, P_n}\right|_{z=0} = 1, \qquad V(0)\, Q_{n+l-2v}^*(0) = \sqrt{R(0)}\, P_n^*(0). \tag{4.7}$$

Remark 4.1. (a) Indeed, in [19] we have even shown a "stronger" version of Theorem 4.1, which holds also if $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ is not necessarily definite and which is the version we applied in the proofs of Theorem 3.1 and Corollary 3.1.

(b) The assumption $\mathscr{L}(x^{\tau}P_n; \mathscr{A}, \mathscr{W}, \lambda) \neq 0$ for a $\tau \in \mathbb{N}$ in Theorem 4.1(b) is equivalent to the fact that the reflection coefficients (a_n) are not all zero from a certain index onward (compare e.g. [20, Theorem 2.1] and Proposition 5.1 below). Thus the equivalence in Theorem 4.1(b) holds if infinitely many reflection coefficients are different from zero, which is always fulfilled since R has only simple zeros by assumption (compare again Proposition 5.1 and Theorem 5.1).

Let now (P_n) be the sequence of orthogonal polynomials with respect to the definite functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$. Then by Theorem 4.1 there exists a uniquely determined sequence of corresponding polynomials $(Q_{n+l-2v})_{n \geqslant a+l/2-w}$. If the P_n's are monic, and this is only a question of normalization, the Q_{n+l-2v}'s fulfill the same recurrence relation as the P_n's:

LEMMA 4.1. *Let* $n_0 := a + l/2 - w$ *and let* $(Q_{n+l-2v})_{n \geqslant n_0}$ *be the sequence of polynomials defined in* (4.4) *associated to the monic orthogonal polynomials* $P_n(z) = z^n + \cdots$ *with respect to* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$. *Then there holds for all* $n \geqslant n_0$

$$Q_{(n+1)+l-2v}(z) = zQ_{n+l-2v}(z) - \bar{a}_n Q^*_{n+l-2v}(z) = Kz^{(n+1)+l-2v} + \cdots, \quad (4.8)$$

where $a_n = -\overline{P_{n+1}(0)}$ (*compare* (1.5)) *and where* K *is the leading coefficient of* Q_{n_0+l-2v}.

Proof. From (4.4) we get

$$Q^*_{n+l-2v}(z) = \frac{\Omega^*_n(z)\, A(z) - P^*_n(z)\, B(z)}{z^{a+l/2-w}}, \qquad n \geqslant n_0$$

and thus by (1.1) and (1.10)

$$\begin{aligned} z^{a+l/2-w} Q_{(n+1)+l-2v}(z) &= -\Omega_{n+1}(z)\, A(z) - P_{n+1}(z)\, B(z) \\ &= z(-\Omega_n(z)\, A(z) - P_n(z)\, B(z)) - \bar{a}_n(\Omega^*_n(z)\, A(z) \\ &\quad - P^*_n(z)\, B(z)) \\ &= z^{a+l/2-w}(zQ_{n+l-2v}(z) - \bar{a}_n Q^*_{n+l-2v}(z)). \quad \blacksquare \end{aligned}$$

We now state our main result which gives, under the assumption that there exists a complex T-polynomial, explicit representations of the orthogonal polynomials with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ and says in particular that the reflection coefficients of these polynomials are (pseudo-) periodic. In this context we call a complex sequence (a_n) (pseudo-)periodic with period $N \in \mathbb{N}$ and preperiod $n_0 \in \mathbb{N}_0$ if it satisfies $a_{n+N} = \mu a_n$ for all $n \geqslant n_0$ with $\mu = 1$ ($|\mu| = 1$).

THEOREM 4.2. *Let the definite functional* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ *be given as in* (2.26) *and suppose that there exists a complex T-polynomial* $\mathscr{T}_N(z) = \alpha z^N + \cdots$, $|\alpha| = 1$, *on* E_l. *Further, let* $P_n(z) = z^n + \cdots$, $n \in \mathbb{N}_0$, *be the monic orthogonal polynomials with respect to* $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ *and let* Q_{n+l-2v}, $n \geqslant a + l/2 - w$, *be given as in* (4.4). *Then there hold with* $n_0 := a + l/2 - w$:

(a) *For all* $v \in \mathbb{N}$ *and* $n \geqslant n_0$

$$\begin{aligned} 2\alpha^v P_{n+vN}(z) &= \mathscr{T}_{vN}(z)\, P_n(z) + V(z)\, \mathscr{U}_{vN-l}(z)\, Q_{n+l-2v}(z) \\ 2\alpha^v Q_{(n+vN)+l-2v}(z) &= \mathscr{T}_{vN}(z)\, Q_{n+l-2v}(z) + W(z)\, \mathscr{U}_{vN-l}(z)\, P_n(z), \end{aligned} \quad (4.9)$$

where $\mathscr{T}_{vN}$ *and* $\mathscr{U}_{vN-l}$ *are given as in Theorem* 3.2.

(b) *The reflection coefficients* (a_n) *of the* P_n*'s satisfy*

$$a_{n+N} = \alpha^2 a_n \qquad \text{for all} \quad n \geqslant n_0. \tag{4.10}$$

Proof. (a) We prove the assertion by applying Theorem 4.1(a). Thus we first have to show that the degrees of the polynomials at the right-hand side in (4.9) are $n + vN$ and $n + vN + l - 2v$, respectively. To see this consider that by (3.8), (4.2), and (4.4) we have

$$\begin{aligned}
&z^{a+l/2-w}[\mathcal{T}_{vN} P_n - V \mathcal{U}_{vN-l} Q_{n+l-2v}]^{(*)}_{n+vN} \\
&\quad = z^{a+l/2-w}[\mathcal{T}_{vN} P_n^* - V \mathcal{U}_{vN-l} Q^*_{n+l-2v}] \\
&\quad = V \mathcal{U}_{vN-l} A[P_n^* F(\cdot; A, W, \lambda) - \Omega_n^*] + \dot{O}(z^{vN+a+l/2-w}) \\
&\quad = O(z^{n+1}) + \dot{O}(z^{vN+a+l/2-w}),
\end{aligned}$$

since $P_n^*(z) F(z; A, W, \lambda) - \Omega_n^*(z) = O(z^{n+1})$ by [9, (18.11)] or by [20, Theorem 2.1], which implies $\mathcal{T}_{vN}(0) P_n^*(0) - V(0) \mathcal{U}_{vN-l}(0) Q^*_{n+l-2v}(0) = 0$. This last identity together with $\mathcal{T}_{vN}(0) P_n^*(0) \neq 0$ gives

$$\begin{aligned}
&[\mathcal{T}_{vN} P_n + V \mathcal{U}_{vN-l} Q_{n+l-2v}]^{(*)}_{n+vN}(0) \\
&\quad = \mathcal{T}_{vN}(0) P_n^*(0) + V(0) \mathcal{U}_{vN-l}(0) Q^*_{n+l-2v}(0) \neq 0
\end{aligned}$$

and thus

$$\partial[\mathcal{T}_{vN} P_n + V \mathcal{U}_{vN-l} Q_{n+l-2v}] = n + vN.$$

The second degree assertion can be shown in the same way.

Now from the two identities

$$\begin{aligned}
&V(\mathcal{T}_{vN} Q_{n+l-2v} + W \mathcal{U}_{vN-l} P_n) \pm \sqrt{R}(\mathcal{T}_{vN} P_n + V \mathcal{U}_{vN-l} Q_{n+l-2v}) \\
&\quad = (\mathcal{T}_{vN} \pm \sqrt{R}\, \mathcal{U}_{vN-l})(V Q_{n+l-2v} \pm \sqrt{R}\, P_n) \\
&V[\mathcal{T}_{vN} Q_{n+l-2v} + W \mathcal{U}_{vN-l} P_n]^* \pm \sqrt{R}[\mathcal{T}_{vN} P_n + V \mathcal{U}_{vN-l} Q_{n+l-2v}]^* \\
&\quad = (\mathcal{T}_{vN} \pm \sqrt{R}\, \mathcal{U}_{vN-l})(V Q^*_{n+l-2v} \pm \sqrt{R}\, P_n^*)
\end{aligned}$$

and from (3.8) and (4.3) it follows that the polynomials at the right-hand side of (4.9) fulfill, in the same way as P_{n+vN} and $Q_{(n+vN)+l-2v}$, a system of the form (4.3). Thus the identities in (4.9) follow from Theorem 4.1(a) and from the definiteness of $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$, where the normalization factor $2\alpha^v$ can be obtained by comparing the leading coefficients with the help of (4.3), (4.8), and (3.8).

(b) By (4.9), (1.1), and (4.8) we have for $n \geqslant n_0$

$$P_{(n+1)+N} = \frac{1}{2\alpha}\left[\mathscr{T}_N(zP_n - \bar{a}_n P_n^*) + V\mathscr{U}_{N-l}(zQ_{n+l-2v} - \bar{a}_n Q_{n+l-2v}^*)\right]$$

$$= zP_{n+N} - \bar{a}_n \frac{\bar{\alpha}}{\alpha} P_{n+N}^* \qquad \text{(again by (4.9))},$$

i.e., $a_{n+N} = (\alpha/\bar{\alpha})\, a_n = \alpha^2 a_n$. ∎

Theorem 4.2(a) gives a tool to generate all the orthogonal polynomials P_n, $n \geqslant n_0 + N$, with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ in an explicit way from the finitely many polynomials P_m and Ω_m, $m = n_0, ..., n_0 + N - 1$. Note that the polynomials Q_{m+l-2v} are given by (4.4) and the sequence of complex T-polynomials $(\mathscr{T}_{vN})_{v \in \mathbb{N}}$ and $(\mathscr{U}_{vN-l})_{v \in \mathbb{N}}$ by Theorem 3.2.

From Theorem 4.2(b) we see that the reflection coefficients of the P_n's orthogonal with respect to $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$ are periodic if there exists a monic complex T-polynomial $\mathscr{T}_N$ on E_l and they are pseudo-periodic if the leading coefficient of $\mathscr{T}_N$ is $\alpha = e^{i\gamma}$, $\gamma \in (0, 2\pi)$. Note that the (pseudo-) periodicity of the reflection coefficients only depends on the set E_l and *not* on A, W, or λ! Roughly speaking, in Section 5 (compare also Theorem 4.4) we will see that there holds also the reversion of this fact; hence, the (pseudo-)periodicity of the reflection coefficients is equivalent to the existence of a complex T-polynomial on a set E_l.

We now give some further recurrence-relations for the case that there exists a complex T-polynomial $\mathscr{T}_N(z) = \alpha z^N + \cdots$ on E_l.

COROLLARY 4.1. *Let the assumptions of Theorem* 4.2 *be fulfilled. Then there hold*:

(a) *For* $v \in \mathbb{N}_0$ *and* $n \geqslant a + l/2 - w$ *we have*

$$P_{(v+2)N+n}(z) = \frac{1}{\alpha}\mathscr{T}_N(z)\, P_{(v+1)N+n}(z) - \frac{L^2 z^N}{4\alpha^2} P_{vN+n}(z)$$

$$Q_{((v+2)N+n)+l-2v}(z) = \frac{1}{\alpha}\mathscr{T}_N(z)\, Q_{((v+1)N+n)+l-2v}(z) - \frac{L^2 z^N}{4\alpha^2} Q_{(vN+n)+l-2v}(z),$$

where the constant L *is given as in* (3.1).

(b) *For* $v \in \mathbb{N}$ *and* $n \geqslant (a + l/2 - w) + vN$ *we have*

$$\frac{L_v^2 z^{vN}}{2\alpha^v} P_{n-vN}(z) = \mathcal{T}_{vN}(z)\, P_n(z) - V(z)\mathcal{U}_{vN-l}(z)\, Q_{n+l-2v}(z)$$

$$\frac{L_v^2 z^{vN}}{2\alpha^v} Q_{(n-vN)+l-2v}(z) = \mathcal{T}_{vN}(z)\, Q_{n+l-2v}(z) - W(z)\, \mathcal{U}_{vN-l}(z)\, P_n(z),$$

where $\mathcal{T}_{vN}$ *and* $\mathcal{U}_{vN-l}$ *are the complex T-polynomials on* E_l *given in Theorem* 3.2 *and* L_v *is the corresponding constant from* (3.1).

Proof. (a) From (4.9) one obtains

$$\begin{aligned} P_{(v+2)N+n} &= \frac{1}{2\alpha}\Big[\mathcal{T}_N P_{(v+1)N+n} + V\mathcal{U}_{N-l}\frac{1}{2\alpha} \\ &\qquad \times (\mathcal{T}_N Q_{(vN+n)+l-2v} + W\mathcal{U}_{N-l}P_{vN+n})\Big] \\ &= \frac{1}{2\alpha}\Big[\mathcal{T}_N P_{(v+1)N+n} + \frac{1}{2\alpha} R\mathcal{U}_{N-l}^2 P_{vN+n} \\ &\qquad + \frac{1}{2\alpha}\mathcal{T}_N(2\alpha P_{(v+1)N+n} - \mathcal{T}_N P_{vN+n})\Big] \\ &= \frac{1}{\alpha}\mathcal{T}_N P_{(v+1)N+n} - \frac{L^2 z^N}{4\alpha^2} P_{vN+n} \qquad \text{(by (3.1))}. \end{aligned}$$

The second identity can be shown in the same way.

(b) Similar to the proof of Theorem 4.2 one obtains that the polynomials at the right-hand sides have zeros at $z=0$ at least of multiplicity vN and a degree of n, resp. $n+l-2v$. Further, the polynomials $(\mathcal{T}_{vN}P_n - V\mathcal{U}_{vN-l}Q_{n+l-2v})/z^{vN}$ and $(\mathcal{T}_{vN}Q_{n+l-2v} - W\mathcal{U}_{vN-l}P_n)/z^{vN}$ satisfy, in the same way as the polynomials P_{n-vN} and $Q_{(n-vN)+l-2v}$, a system of the form (4.3) and thus they coincide up to a constant factor with P_{n-vN} and $Q_{(n-vN)+l-2v}$ by Theorem 4.1(a) and the definiteness of $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$. From (4.8), (4.9) (writing $P_n = P_{(n-vN)+vN}$) and (3.1) one can derive that the leading coefficient of the polynomials at the right-hand sides is $L_v^2/2\alpha^v$, resp. $KL_v^2/2\alpha^v$ (note Lemma 4.1), and the assertion follows. ∎

The next corollary gives explicit representations of the polynomials $g_{(n)}$ from (4.5). As we will show in a forthcoming paper, these polynomials play an important role in describing the associated polynomials, defined in (1.12) and (1.13).

COROLLARY 4.2 *Let the assumptions of Theorem 4.2 be fulfilled and let $p \in \mathbb{N}_0$ be the order of the zero of P_n at $z=0$. Then for $n \geqslant a+l/2+v$ the polynomial $g_{(n)}$, given as in (4.5), can be represented as*

$$g_{(n)}(z) = 2\alpha\, d_n \frac{P_N^{(n)}(z) - \Omega_N^{(n)}(z)}{z^p \mathscr{U}_{N-l}(z)}, \qquad d_n \text{ from } (1.11). \tag{4.11}$$

Further, there holds

$$g_{(n+N)}(z) = \frac{d_N^{(n)}}{\alpha^2}\, g_{(n)}(z), \qquad \text{where} \quad d_N^{(n)} := \prod_{j=0}^{N-1} (1 - |a_{n+j}|^2). \tag{4.12}$$

Proof. With the help of (4.5) and (4.9) we get

$$\begin{aligned} 2\alpha(Q_{(n+N)+l-2v} P_n - Q_{n+l-2v} P_{n+N}) &= \mathscr{U}_{N-l}(W P_n^2 - V Q_{n+l-2v}^2) \\ &= \mathscr{U}_{N-l} z^{n+p-(a+l/2-w)} A g_{(n)}. \end{aligned}$$

On the other hand, we have by (4.4)

$$z^{a+l/2-w}(Q_{(n+N)+l-2v} P_n - Q_{n+l-2v} P_{n+N}) = A(P_{n+N}\Omega_n - P_n \Omega_{n+N}).$$

From these two identities there follows

$$\mathscr{U}_{N-l}\, g_{(n)} = \frac{2\alpha}{z^{n+p}}(P_{n+N}\Omega_n - P_n\Omega_{n+N}) = \frac{2\alpha\, d_n}{z^p}(P_N^{(n)} - \Omega_N^{(n)}),$$

where we have used (1.11), (1.15), and (1.16) for the last identity.

In order to prove (4.12) let us consider that by Theorem 4.2(b) the polynomials $P_N^{(n)}$, $\Omega_N^{(n)}$ are generated by the reflection coefficients $a_n, ..., a_{n+N-1}$ and the polynomials $P_N^{(n+N)}$, $\Omega_N^{(n+N)}$ by $\alpha^2 a_n, ..., \alpha^2 a_{n+N-1}$, where $|\alpha| = 1$. From [9, (7.4) and (7.9)] we have the relations

$$P_N^{(n+N)} = \frac{1+\bar{\alpha}^2}{2} P_N^{(n)} + \frac{1-\bar{\alpha}^2}{2} \Omega_N^{(n)}$$

and

$$\Omega_N^{(n+N)} = \frac{1-\bar{\alpha}^2}{2} P_N^{(n)} + \frac{1+\bar{\alpha}^2}{2} \Omega_N^{(n)}$$

and thus

$$P_N^{(n+N)} - \Omega_N^{(n+N)} = \frac{1}{\alpha^2}(P_N^{(n)} - \Omega_N^{(n)}).$$

Now the assertion follows from (4.11); note thereby that $g_{(n)}(0) \neq 0$ and $g_{(n+N)}(0) \neq 0$. ∎

Let us point out that in what follows it suffices to consider the case of a monic complex T-polynomial $\mathcal{T}_N(z) = z^N + \cdots$, i.e., $\alpha = 1$.

Remark 4.2. Let the set $E_l = \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}]$, the polynomials R, W, A, and the functional $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$ be given. Then, choosing an arbitrary $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, the functional $\mathcal{L}(\cdot; \hat{\mathcal{A}}, \hat{\mathcal{W}}, \lambda)$ is again of the form (2.26), where now

$$\hat{E}_l := \bigcup_{j=1}^{l} [\hat{\varphi}_{2j-1}, \hat{\varphi}_{2j}], \qquad \hat{\varphi}_j := \varphi_j + \arg d, \qquad \text{where} \quad d := \alpha^{2/N}$$

$$\hat{R}(x) := d^l R\left(\frac{z}{d}\right), \qquad \hat{A}(z) := d^a A\left(\frac{z}{d}\right), \qquad \hat{W}(z) := d^w W\left(\frac{z}{d}\right).$$

In Remark 3.1 we have shown that there exists a complex T-polynomial $\mathcal{T}_N(z) = \alpha z^N + \cdots$ on E_l if and only if there exists a monic complex T-polynomial $\hat{\mathcal{T}}_N(z) = z^N + \cdots$ on $\hat{E}_l$. Further, the monic orthogonal polynomials (P_n) with respect to $\mathcal{L}(\cdot; \mathcal{A}, \mathcal{W}, \lambda)$, generated by the reflection coefficients (a_n), and the monic orthogonal polynomials $(\hat{P}_n)$ with respect to $\mathcal{L}(\cdot; \hat{\mathcal{A}}, \hat{\mathcal{W}}, \lambda)$, generated by the reflection coefficients $(\hat{a}_n)$, are related by

$$\hat{P}_n(z) = d^n P_n\left(\frac{z}{d}\right) = z^n + \cdots, \qquad n \in \mathbb{N}_0,$$

and there holds

$$\hat{a}_n = d^{-(n+1)} a_n, \qquad n \in \mathbb{N}_0,$$

where now by Theorem 4.2(b) $\hat{a}_{n+N} = \hat{a}_n$ for all $n \geqslant a + l/2 - w$.

In order to prove Theorem 4.3 we first need the following lemma, which follows by quite straightforward calculations from (1.15)–(1.18).

LEMMA 4.2. *Let (a_n) be a sequence of periodic reflection coefficients, $|a_n| \neq 1$, with period $N \in \mathbb{N}$ and preperiod $n_0 \in \mathbb{N}_0$. Further, let the polynomials $\tilde{R}_{(n)}$, $\tilde{A}_{(n)}$, $\tilde{B}_{(n)}$, $n \geqslant n_0$, be given as in* (2.6). *Then we have the identities*

$$\tilde{R}_{(n)} = d_n^2 z^{2n} \left[(P_N^{(n)} + \Omega_N^{(n)} + P_N^{(n)*} + \Omega_N^{(n)*})^2 - 16 z^N \prod_{j=0}^{N-1} (1 - |a_{n+j}|^2) \right]$$

$$\tilde{A}_{(n)} = P_n^2 (P_N^{(n)*} - \Omega_N^{(n)*}) + P_n P_n^* (P_N^{(n)*} + \Omega_N^{(n)*} - P_N^{(n)} - \Omega_N^{(n)}) + P_n^{*2} (\Omega_N^{(n)} - P_N^{(n)})$$

$$\tilde{B}_{(n)} = P_n^* \Omega_n^* (\Omega_N^{(n)} - P_N^{(n)}) + P_n \Omega_n (\Omega_N^{(n)*} - P_N^{(n)*}) + \tfrac{1}{2}(P_N^{(n)} + \Omega_N^{(n)} - P_N^{(n)*} - \Omega_N^{(n)*}) \times (\Omega_n P_n^* - \Omega_n^* P_n)$$

$$P_N^{(n)} + \Omega_N^{(n)} + P_N^{(n)*} + \Omega_N^{(n)*} = P_N^{(\nu)} + \Omega_N^{(\nu)} + P_N^{(\nu)*} + \Omega_N^{(\nu)*}, \quad \text{for all} \quad \nu \geqslant n_0.$$

The following theorem gives now explicit representations of the polynomials R, A, $B(\cdot; A, W, \lambda)$ and $\mathcal{T}_N$, where $\mathcal{T}_N$ is monic, in terms of orthogonal polynomials (compare [8, Theorem X] and our considerations in Section 2).

THEOREM 4.3. *Let the assumptions of Theorem* 4.2, *where we now suppose that* $\alpha = 1$, *be fulfilled and let* $B := B(\cdot; A, W, \lambda)$ *be the polynomial given in* (4.2). *Then, with* $\tilde{R}_{(n)}$, $\tilde{A}_{(n)}$, $\tilde{B}_{(n)}$ *from* (2.6) *and* d_n *from* (1.11), *there hold for all* $n, \nu \geqslant n_0 := a + l/2 - w$,

$$R\mathcal{U}_{N-l}^2 = \frac{\tilde{R}_{(n)}}{4d_n^2 z^{2n}} = \frac{1}{4}\left[(P_N^{(\nu)} + \Omega_N^{(\nu)} + P_N^{(\nu)*} + \Omega_N^{(\nu)*})^2 - 16 z^N \prod_{j=\nu}^{\nu+N-1} (1 - |a_j|^2)\right]$$

$$V\mathcal{U}_{N-l} A = \frac{\tilde{A}_{(n)}}{2d_n z^{n-n_0}} = \frac{1}{2d_{n_0}} [P_{n_0}^2 (P_N^{(n_0)*} - \Omega_N^{(n_0)*}) + P_{n_0} P_{n_0}^* (P_N^{(n_0)*} + \Omega_N^{(n_0)*} - P_N^{(n_0)} - \Omega_N^{(n_0)}) + P_{n_0}^{*2} (\Omega_N^{(n_0)} - P_N^{(n_0)})]$$

$$V\mathcal{U}_{N-l} B = \frac{\tilde{B}_{(n)}}{2d_n z^{n-n_0}} = \frac{1}{2d_{n_0}} [P_{n_0}^* \Omega_{n_0}^* (\Omega_N^{(n_0)} - P_N^{(n_0)}) + P_{n_0} \Omega_{n_0} (\Omega_N^{(n_0)*} - P_N^{(n_0)*}) + \frac{1}{2}(P_N^{(n_0)} + \Omega_N^{(n_0)} - P_N^{(n_0)*} - \Omega_N^{(n_0)*})(\Omega_{n_0} P_{n_0}^* - \Omega_{n_0}^* P_{n_0})]$$

$$\mathcal{T}_N = \frac{1}{2d_n z^n} (P_n^* \Omega_{n+N} + \Omega_n^* P_{n+N} + \Omega_n P_{n+N}^* + P_n \Omega_{n+N}^*) = \frac{1}{2}(P_N^{(\nu)} + \Omega_N^{(\nu)} + P_N^{(\nu)*} + \Omega_N^{(\nu)*})$$

$$L^2 = 4 \prod_{j=\nu}^{\nu+N-1} (1 - |a_j|^2).$$

Proof. From the characterization Theorem 2.1 in [20], which characterizes orthogonal polynomials with respect to a linear functional $\mathscr{L}$ with the help of the Stieltjes transform $F(z)=\mathscr{L}((x+z)/(x-z))$, one can obtain by using (1.11), (1.17), and (1.18) that a representation of the form (1.14) even holds in the definite case $|a_n| \neq 1$. Hence, by Theorem 4.2(b), (2.5)–(2.8), and (4.2) we have the identity

$$\frac{V\mathscr{U}_{N-l}B + z^{a+l/2-w}\sqrt{R}\,\mathscr{U}_{N-l}}{V\mathscr{U}_{N-l}A} = \frac{\pm(1/2d_{s_0})\,\tilde{B}_{(s_0)} + z^{s_0}\sqrt{\tilde{R}_{(s_0)}/4d_{s_0}^2 z^{2s_0}}}{\pm(1/2d_{s_0})\,\tilde{A}_{(s_0)}}, \tag{4.13}$$

where s_0 denotes the exact preperiod of the period sequence (a_n), i.e., $a_{s_0+N-1} \neq a_{s_0-1}$ ($a_{-1} := 1$ if $s_0 = 0$); note that $\tilde{R}_{(s_0)}$ has a zero at $z=0$ of exact order $2s_0$ and leading coefficient $4d_{s_0}^2$ by Lemma 4.2. Considering this equation for $z=e^{i\varphi}$, $\varphi \in E_l$, and calculating the real parts one obtains (compare also the proof of Lemma 3.4 in [19])

$$\frac{z^{a+l/2-w}\sqrt{R}\,\mathscr{U}_{N-l}}{V\mathscr{U}_{N-l}A} = \frac{z^{s_0}\sqrt{\tilde{R}_{(s_0)}/4d_{s_0}^2 z^{2s_0}}}{\pm(1/2d_{s_0})\,\tilde{A}_{(s_0)}}. \tag{4.14}$$

Since the polynomials R, V, $\mathscr{U}_{N-l}$, A, $\tilde{R}_{(s_0)}/z^{2s_0}$, and $\tilde{A}_{(s_0)}$ do not vanish at $z=0$ by definition and by (2.6), compare also Lemma 4.2, we have

$$n_0 = a + \frac{l}{2} - w = s_0 \qquad \text{is the exact preperiod.} \tag{4.15}$$

In what follows we will write n_0 instead of s_0. By (4.14) and (4.15) the polynomial $\tilde{R}_{(n_0)}/z^{2n_0}$ must vanish at the zeros of R and there are no other square-root zeros at the right-hand side, i.e., we can write $\tilde{R}_{(n_0)}/4d_{n_0}^2 z^{2n_0} =: R\tilde{\mathscr{U}}^2$, where $\tilde{\mathscr{U}} \in \mathbb{P}_{N-l}$. By the representation of $\tilde{R}_{(n_0)}$ in Lemma 4.2 and by Theorem 3.1(a), i.e., the uniqueness of complex T-polynomials, there follows that $\mathscr{U} = \mathscr{U}_{N-l}$. Thus we get

$$R\mathscr{U}_{N-l}^2 = \frac{\tilde{R}_{(n_0)}}{4d_{n_0}^2 z^{2n_0}}, \tag{4.16}$$

(note that by (3.1) and Lemma 4.2 both sides in (4.16) are monic polynomials), which is by Lemma 4.2 and (2.7) the first assertion of the theorem. As a consequence there even follows the representation of the complex T-polynomial $\mathscr{T}_N$, compare also (1.15)–(1.18), and of the constant L. With the help of (4.13), (4.15), and (4.16) we further get

$$V\mathscr{U}_{N-l}A = \pm\frac{1}{2d_{n_0}}\,\tilde{A}_{(n_0)} \qquad \text{and} \qquad V\mathscr{U}_{N-l}B = \pm\frac{1}{2d_{n_0}}\,\tilde{B}_{(n_0)}. \tag{4.17}$$

Again by Lemma 4.2 and (2.7) it remains to show that these identities hold with the positive sign.

By (4.9) we can write

$$\tilde{Q}_{n+N} := V\mathscr{U}_{N-l} Q_{n+l-2v} = 2P_{n+N} - \mathscr{T}_N P_n, \qquad n \geqslant n_0 + N.$$

Using the representations of $R\mathscr{U}_{N-l}^2$ and $\mathscr{T}_N$ in this theorem we get after some tedious but straightforward calculation with the help of the identities given in (1.11), (1.15), (3.1) and Lemma 4.2 that

$$\begin{aligned} \mathscr{U}_{N-l}^2 RP_n^2 - \tilde{Q}_{n+N}^2 &= z^{n-n_0} \frac{1}{2d_{n_0}} \tilde{A}_{(n_0)} \cdot 2d_n (P_N^{(n)} - \Omega_N^{(n)}) \\ &= z^{n+p-n_0} \frac{1}{2d_{n_0}} \tilde{A}_{(n_0)} \cdot \mathscr{U}_{N-l} g_{(n)} \qquad \text{(by Corollary 4.2).} \end{aligned}$$

Comparing this equality with (4.5), i.e., $\mathscr{U}_{N-l}^2 RP_n^2 - \tilde{Q}_{n+N}^2 = z^{n+p-n_0} V\mathscr{U}_{N-l} A \cdot \mathscr{U}_{N-l} g_{(n)}$, and recalling Remark 4.1(b), the first positive sign in (4.17) follows and by (4.13) also the second one. ∎

By combining Theorem 4.2 and Theorem 4.3 with the result of Geronimus [8, Theorem X] mentioned in the Introduction, we get

THEOREM 4.4. *Let (a_n) with $|a_n| < 1$ be the reflection coefficients of the orthogonal polynomials (P_n). Then (a_n) is periodic from a certain index n_0 onward if and only if (P_n) is orthogonal with respect to a definite functional of the form* (2.26) *and on E_l there exists a monic complex T-polynomial.*

Proof. The necessity part follows from [8, Theorem X]—recall also our statements in Section 2—where the fact that there exists a monic complex T-polynomial can be derived from Theorem 4.3 (see also the assertions of the following Section 5: note that either $a_n = 0$ for all $n \geqslant n_0$, which is the situation of Proposition 5.1 below, or infinitely many a_n satisfy $0 < |a_n| < 1$; in the second case all the assumptions (2.16)–(2.21) follow from the positive definiteness of the orthogonality measure). The sufficiency part is Theorem 4.2. ∎

Let us recall that (P_n) orthogonal with respect to a positive definite functional of the form (2.26) means that (P_n) is orthogonal with respect to the weight function $\sqrt{-\mathscr{R}(\varphi)/\mathscr{A}(\varphi)} +$ eventually point measures at the zeros of $\mathscr{A}$.

5. DEFINITE FUNCTIONALS ASSOCIATED WITH PERIODIC REFLECTION COEFFICIENTS

In this section we determine all definite functionals with respect to which polynomials generated by periodic reflection coefficients are orthogonal.

Let us recall that by the definiteness of the functional the reflection coefficients (a_n) satisfy $|a_n| \neq 1$ for all $n \in \mathbb{N}_0$. The positive definite case $|a_n| < 1$ (with the additional assumption $|a_n| > 0$) has been treated and solved by Geronimus [8].

Since in the following we will have to distinguish between the case that $a_n = 0$ for $n \geqslant n_0$ and $a_n \neq 0$ for infinitely many n (compare e.g. Remark 4.1(b)) let us consider some aspects of the first case, which lead to orthogonal polynomials on the whole unit circle, which can be considered as a generalization of the Bernstein–Szegő polynomials.

PROPOSITION 5.1. *Let* $P_{n_0}(z) := \prod_{j=1}^{q} (z - z_j) \prod_{j=q+1}^{n_0} (z - z_j)$, *where* $|z_j| > 1$ *for* $j = 1, ..., q$ *and* $|z_j| < 1$ *for* $j = q+1, ..., n_0$, *and let the trigonometric polynomial* $\mathscr{A}$ *be given by*

$$\mathscr{A}(\varphi) := e^{-in_0\varphi} P_{n_0}(e^{i\varphi})\, P^*_{n_0}(e^{i\varphi}) = |P_{n_0}(e^{i\varphi})|^2. \tag{5.1}$$

Then the polynomials

$$P_n(z) := z^{n-n_0} P_{n_0}(z), \qquad n \geqslant n_0, \tag{5.2}$$

have the 2π *orthogonality property*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} P_n(e^{i\varphi}) \frac{d\varphi}{\mathscr{A}(\varphi)} - \sum_{j=1}^{q} \operatorname{res}\left(\frac{z^{n-k-1}}{P^*_{n_0}(z)}, z_j\right) = 0$$
$$\text{for} \quad k = \ldots, -2, -1, 0, 1, ..., n-1. \tag{5.3}$$

Proof. Since for $k = \ldots, -2, -1, 0, 1, ..., n-1$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} (e^{i(n-n_0)\varphi} P_{n_0}(\varphi)) \frac{d\varphi}{\mathscr{A}(\varphi)} = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{n-k-1}}{P^*_{n_0}(z)}\, d\varphi$$
$$= \sum_{j=1}^{q} \operatorname{res}\left(\frac{z^{n-k-1}}{P^*_{n_0}(z)}, z_j\right),$$

where the last equality follows by the Residuum Theorem, the assertion is proved. ∎

Let us note that (5.2) implies by (1.1) that

$$a_{n_0} = a_{n_0+1} = \cdots = 0. \tag{5.4}$$

If P_{n_0} has all its zeros in $|z| < 1$, i.e., $q = 0$, then the polynomials P_n are the well-known Bernstein–Szegő polynomials (see e.g. [4; 24, p. 31]).

Concerning the connection to our functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{W}, \lambda)$, let us point out that the orthogonality condition (5.3) is nothing other than

$$\mathscr{L}(x^{-k}P_n; \mathscr{A}, \mathscr{W}, \lambda) = 0 \qquad \text{for} \quad k = ..., -2, -1, 0, 1, ..., n-1, \tag{5.5}$$

where

$$E_1 = [0, 2\pi], \qquad R(z) = (z-1)^2, \qquad W(z) = i(1-z), \qquad \mathscr{A} \text{ defined in (5.1)}$$

and λ given by, with $j \in \{1, ..., n_0\}$,

$$\lambda_j = \begin{cases} +1 & \text{if} \quad |z_j| < 1, \quad \text{i.e.,} \quad j \in \{q+1, ..., n_0\} \\ -1 & \text{if} \quad |z_j| > 1, \quad \text{i.e.,} \quad j \in \{1, ..., q\} \end{cases}.$$

Thus the orthogonality property (5.5) is a direct consequence of our general results [19, Theorem 2.2 and Remark 2.3], but the above given proof seems to be worth mentioning.

Now we consider the general case of reflection coefficients (a_n) satisfying for a $n_0 \in \mathbb{N}_0$, $N \in \mathbb{N}$, and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$,

$$a_{n+N} = \alpha^2 a_n \qquad \text{for} \quad n \geqslant n_0 \qquad \text{and} \qquad |a_n| \neq 1, \quad n \in \mathbb{N}_0, \tag{5.6}$$

where n_0 is the exact preperiod, i.e.,

$$a_{n_0-1} \neq a_{n_0-1+N} \qquad (a_{-1} := 1 \qquad \text{if} \quad n_0 = 0). \tag{5.7}$$

Note that we also allow the special case (5.4), treated in Proposition 5.1. As usual, we denote by P_n and Ω_n the monic orthogonal polynomials, resp. the polynomials of the second kind generated by (1.1) and (1.10) with this sequence (a_n). Then we define the following polynomials

$$\mathscr{T}_N := \frac{1}{2}(\alpha P_N^{(n_0)} + \alpha \Omega_N^{(n_0)} + \bar{\alpha} P_N^{(n_0)*} + \bar{\alpha} \Omega_N^{(n_0)*}) = \alpha z^N + \cdots + \bar{\alpha} \tag{5.8}$$

$$\mathfrak{R} := \mathscr{T}_N^2 - 4z^N \prod_{j=0}^{N-1} (1 - |a_{n_0+j}|^2) = \alpha^2 z^{2N} + \cdots + \bar{\alpha}^2 \tag{5.9}$$

$$\mathfrak{A} := \frac{1}{d_{n_0}} (\bar{\alpha} P_{n_0} P_{n_0+N}^* - \alpha P_{n_0}^* P_{n_0+N}) \tag{5.10}$$

$$\mathfrak{B} := \frac{1}{2d_{n_0}} (\alpha P_{n_0}^* \Omega_{n_0+N} - \alpha \Omega_{n_0}^* P_{n_0+N} - \bar{\alpha} \Omega_{n_0} P_{n_0+N}^* + \bar{\alpha} P_{n_0} \Omega_{n_0+N}^*). \tag{5.11}$$

Finally, we define the function

$$\mathfrak{F}(z) := \frac{\mathfrak{B}(z) + z^{n_0} \sqrt{\mathfrak{R}(z)}}{\mathfrak{A}(z)}. \tag{5.12}$$

By (5.7), (5.9), and (5.10) we have $\mathfrak{R}(0) \neq 0$ and $\mathfrak{A}(0) \neq 0$, thus $\mathfrak{F}$ is analytic at $z=0$ and can be expanded in a power series

$$\mathfrak{F}(z) =: \mathfrak{c}_0 + 2 \sum_{j=1}^{\infty} \mathfrak{c}_j z^j.$$

Let $\mathfrak{L}$ be the corresponding linear functional defined as in (1.3) with the moment sequence $(\mathfrak{c}_j)$, i.e.,

$$\mathfrak{L}(x^{-j}) = \mathfrak{c}_j \qquad \text{and} \qquad \mathfrak{c}_j = \overline{\mathfrak{c}_{-j}}, \qquad j \in \mathbb{Z}. \tag{5.13}$$

Then there holds

THEOREM 5.1. *Let* (P_n) *and the polynomials of the second kind* (Ω_n) *be generated by* (1.1) *and* (1.10), *respectively, with the periodic sequence of reflection coefficients* (a_n), *given as in* (5.6). *Then the* P_n*'s are orthogonal with respect to the functional* $\mathfrak{L}$ *from* (5.13), *i.e.*,

$$\mathfrak{L}(x^{-j} P_n) = 0 \qquad \textit{for} \quad j = 0, ..., n-1. \tag{5.14}$$

Proof. Let us first assume the more interesting case that infinitely many a_n's are unequal to zero. Then we prove the theorem by using Theorem 2.1 in [20], which is a general characterization theorem for orthogonal polynomials with respect to a linear functional of the form (1.3) and which says that the orthogonality property (5.14) is equivalent to the system

$$\begin{cases} P_n(z)\, \mathfrak{F}(z) + \Omega_n(z) = O(z^n) \\ P_n^*(z)\, \mathfrak{F}(z) - \Omega_n^*(z) = O(z^{n+1}), \end{cases} \qquad \text{as} \quad z \to 0. \tag{5.15}$$

We show that the conditions in (5.15) are fulfilled for all $n \geqslant n_0 + N$, i.e., $(P_n)_{n \geqslant n_0 + N}$ are orthogonal polynomials with respect to $\mathfrak{F}$. The orthogonality properties of the P_n's, $n = 0, ..., n_0 + N - 1$, follow for example from [9, Theorem 4.1 and Theorem 6.1].

Let us define

$$Q_{n+N} := 2\alpha P_{n+N} - \mathcal{T}_N P_n.$$

As in the proof of Theorem 4.3 one can show that

$$\mathfrak{R}(z)\, P_n^2(z) - Q_{n+N}^2(z) = z^{n+p-n_0} \mathfrak{A}(z)\, g_{(n)}(z), \qquad n \geqslant n_0 + N, \tag{5.16}$$

where p denotes the order of the zero of P_n at $z=0$ and where (compare with Corollary 4.2)

$$g_{(n)}(z)=2\alpha d_n(P^{(n)}_{N-p}-\Omega^{(n)}_{N-p})\in\mathbb{P}_{N-p-1}, \qquad g_{(n)}(0)\neq 0.$$

Further, one gets by some tedious but straightforward calculation that

$$Q_{n+N}(z)=-\frac{\Omega_n(z)\,\mathfrak{A}(z)+P_n(z)\,\mathfrak{B}(z)}{z^{n_0}}, \qquad n\geqslant n_0, \tag{5.17}$$

and

$$\left.\frac{Q_{n+N}}{\sqrt{\mathfrak{R}}\,P_n}\right|_{z=0}=\frac{Q^*_{n+N}(0)}{\sqrt{\mathfrak{R}(0)}\,P^*_n(0)}=1. \tag{5.18}$$

Now the system (5.15) follows from (5.16)–(5.18) by comparing coefficients (for this compare also our proofs of Theorem 2.2 and Theorem 3.10 in [19]).

If $a_{n_0+j}=0$ for all $j\in\mathbb{N}_0$ then we are in the situation of Proposition 5.1 and we have $N=1$ and $\alpha=1$. Further, $\mathfrak{R}(z)=(1-z)^2$, $\mathfrak{A}(z)=(1/d_{n_0})(1-z)\,P_{n_0}(z)\,P^*_{n_0}(z)$, $\mathfrak{B}(z)=(1/2d_{n_0})(1-z)(\Omega^*_{n_0}(z)\,P_{n_0}(z)-\Omega_{n_0}(z)\,P^*_{n_0}(z))$, and with the help of (1.11) we get

$$\mathfrak{F}(z)=\frac{\Omega^*_{n_0}(z)}{P^*_{n_0}(z)},$$

which is the right Stieltjes transform by [20, Theorem 2.1]; see also [19, Remark 2.3(c)] (for the orthogonality properties of the lower degree polynomials P_n, $n=0, ..., n_0-1$, again see e.g. the Proof of Theorem 6.1 and Theorem 4.1 [9]). ∎

Note that the polynomials $\mathfrak{R}$ and $\mathfrak{A}$ need not satisfy all the conditions (2.16)–(2.20) and thus in general the function $\mathfrak{F}$, defined in (5.12), is not necessarily of the form (4.2). For example, if we consider the constant sequence of reflection coefficients (a_n) with $|a|>1$ then the polynomial $\mathfrak{R}$ from (5.9) is of the form $\mathfrak{R}(z)=(1-z)^2+4\,|a|^2\,z$ and has zeros outside the unit circumference. On the other hand, the sequence (a_n) with $a_0=1+i$, $a_1=2-3i$, and $a_n=\frac{1}{2}$ for $n\geqslant 2$ yields polynomials $\mathfrak{R}$ and $\mathfrak{A}$, which satisfy (2.15)–(2.21) for the set $E_1=[1.047, 5.236]$, although $|a_0|$, $|a_1|>1$.

Now the following question arises. Given a union of intervals E_l on which there exists a complex T-polynomial, how many periodic sequences of reflection coefficients are generated by polynomials orthogonal on E_l with respect to nonnegative weight functions of the form $|\sqrt{\mathcal{R}(\varphi)}/\mathcal{A}(\varphi)|$?

In the case of the whole unit circumference, i.e., $E_l = E_1 = [0, 2\pi]$, we know that there is only one such periodic sequence, the sequence $a_n = 0$ for $n \in \mathbb{N}_0$. But as we shall see this is a real exceptional case because in all other cases there is an infinite set of periodic sequences. Let us demonstrate this by the following example.

EXAMPLE 5.1. (a) Let (P_n) be generated by the sequence of reflection coefficients

$$a_n := a \qquad \text{for} \quad n \in \mathbb{N}_0, \qquad \text{with} \quad |a| < 1.$$

Then by Theorem 5.1 (P_n) is orthogonal with respect to the positive functional $\mathscr{L}(\cdot; \mathscr{A}, \mathscr{R}, \lambda)$, where

$$\mathscr{T}_1(z) = z + 1, \qquad R(z) = \mathscr{T}_1^2(z) - 4(1 - |a|^2)\, z,$$
$$A(z) = (1 + \bar{a}) - (1 + a)\, z, \qquad B(z) = \bar{a} + az$$

(possible common zeros of R and A lead to the splitting $R = VW$). Hence (take a look at R and recall that $\mathscr{T}_1$ is independent of a), if we fix a $\gamma \in (0, 1)$ we obtain that for every

$$a \in \mathscr{S}^1 := \{a \in \mathbb{C} : |a| = \gamma\}$$

the polynomials (P_n) generated by $a_n = a$, $n \in \mathbb{N}_0$, are orthogonal on the same interval

$$[\alpha, 2\pi - \alpha] + \text{eventually a point measure at } \frac{1 + \bar{a}}{1 + a}$$

where

$$e^{\pm i\alpha} := 1 - 2\gamma^2 \pm 2i\gamma\sqrt{1 - \gamma^2}.$$

(b) The case of two intervals can also be treated without problems. In fact, let

$$a_{2n} := a_0 \quad \text{and} \quad a_{2n+1} := a_1 \quad \text{with } |a_0| < 1 \quad \text{and} \quad |a_1| < 1. \tag{5.19}$$

Then by (5.8)–(5.11),

$$\begin{aligned}
&\mathscr{T}_2(z) = z^2 + 2\,\mathrm{Re}\{a_1 \bar{a}_0\}\, z + 1, \\
&R(z) := \mathfrak{R}(z) = \mathscr{T}_2^2(z) - 4(1 - |a_0|^2)(1 - |a_1|^2), \\
&A(z) := \mathfrak{A}(z) = (1 + \bar{a}_1) - 2i(\mathrm{Im}\{a_0\} + \mathrm{Im}\{a_1 \bar{a}_0\})\, z - (1 + a_1)\, z^2, \\
&B(z) := \mathfrak{B}(z) = a_1 z^2 + 2\,\mathrm{Re}\{a_0\}\, z + \bar{a}_1.
\end{aligned} \tag{5.20}$$

Thus for given $\gamma_1, \gamma_2 \in (0, 1)$ we obtain that for every $(a_0, a_1) \in \mathscr{S}^2$, where

$$\mathscr{S}^2 := \{(a_0, a_1) \in \mathbb{C} \times \mathbb{C}: |a_0| < 1, |a_1| < 1, (1 - |a_0|^2)(1 - |a_1|^2) = \gamma_1 \quad \text{and} \quad \operatorname{Re}\{a_1 \bar{a}_0\} = \gamma_2\},$$

the polynomials (P_n) generated by a sequence of the form (5.19) with $(a_0, a_1) \in \mathscr{S}^2$ are by (5.20) all orthogonal on the same set of two intervals $E_2 = \{\varphi \in [0, 2\pi]: e^{-2i\varphi} R(e^{i\varphi}) \leqslant 0\}$, where at the zeros of A point measures may appear.

In view of the above example we see that there is an infinite set of periodic sequences associated with one system of intervals E_l in contrast to the case of the whole unit circumference. This has consequences for asymptotic considerations because if we want to get asymptotic results for polynomials $p_n(\cdot; w)$ orthogonal on E_l with respect to the weight function w with the help of polynomials orthogonal on E_l with respect to $\sqrt{-\mathscr{R}/\mathscr{A}}$ we will have to look first to which sequence of periodic reflection coefficients the reflection coefficients of the $p_n(\cdot; w)$'s will converge. Naturally this problem does not appear if one considers asymptotics of orthogonal polynomials with limit periodic reflection coefficients since the corresponding periodic sequence is given in advance.

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